

# Photon mixing in universes with large extra–dimensions

Cédric Deffayet<sup>1</sup> and Jean–Philippe Uzan<sup>1,2</sup>

(1) *Laboratoire de Physique Théorique\*, Université Paris XI, Bât. 210  
F–91405 Orsay Cedex (France).*

(2) *Département de Physique Théorique, Université de Genève,  
24 quai E. Ansermet, CH–1211 Genève 4 (Switzerland).*

(February 1, 2008)

In presence of a magnetic field, photons can mix with any particle having a two–photon vertex. In theories with large compact extra–dimensions, there exists a hierarchy of massive Kaluza–Klein gravitons that couple to any photon entering a magnetic field. We study this mixing and show that, in comparison with the four dimensional situation where the photon couples only to the massless graviton, the oscillation effect may be enhanced due to the existence of a large number of Kaluza–Klein modes. We give the conditions for such an enhancement and then investigate the cosmological and astrophysical consequences of this phenomenon; we also discuss some laboratory experiments. Axions also couple to photons in the same way; we discuss the effect of the existence of bulk axions in universes with large extra–dimensions. The results can also be applied to neutrino physics with extra–dimensions.

**Preprint numbers:** UGVA–DPT 99/10-1053, LPT–ORSAY 99/104

## I. INTRODUCTION

It is well known [1,2] that photons can be converted into gravitons by a magnetic field in a standard four dimensional spacetime. The propagation eigenstates are then mixtures of photon and graviton interaction eigenstates. In quantum words, this mixing is due to the fact that any particle which has a two–photon vertex can be created by a photon entering an external electromagnetic field [3,4] and can oscillate coherently with the photon. Classically, this can be understood by the fact that an electromagnetic plane wave cannot radiate gravitationally in vacuum since its stress–energy tensor contains no quadrupole [5]. But, a time varying quadrupole appears (due to interference) when an electromagnetic plane wave propagates through a constant magnetic field [1,2,6]. The implications of this effect on the cosmic microwave background (photons) has been considered [6,7] and it has been shown that it will be undetectable for standard cosmological magnetic fields [8]. A similar effect also generically happens for axions (and for any particle having a two–photon vertex). The photon–axion (see [9–11] for reviews on axions) mixing has yet been studied in details by many authors (see e.g. [4,12–14]) and is used in experiments, since the pioneer work by Sikivie [12], to put constraints on the axion parameters [15–18] (see e.g. [19] for an up to date review on such experiments).

Recently, a lot of interest has been raised by models where the universe has large extra–dimensions [20,21]. In such models, the Planck scale,  $M_4$ , is no longer a fundamental scale but is related to the fundamental mass scale of the  $D$  dimensional theory,  $M_D$ , through [21]

$$\bar{M}_4^2 \equiv R^n M_D^{n+2}, \quad (1)$$

where  $R$  is a length scale (usually taken to be the radius of the  $n = D - 4$  compact extra–dimensions) and  $\bar{M}_4 \equiv M_4/\sqrt{8\pi}$ .  $M_D$  can be significantly smaller than  $\bar{M}_4$  at the price of having large extra–dimensions. These ideas can be naturally embedded in fundamental string theories with a low string scale [22–25] (see also [26] for earlier discussions on TeV scale extra–dimensions). In these models, gravity can propagate in the  $D$  dimensional spacetime (bulk space time) whereas the standard model fields are localised on a 3–brane. An effect of the compact extra–dimensions arises from interactions between the Kaluza–Klein (KK) excitations of the gravitons (or other bulk fields) which are seen in four dimensions as a tower of massive particles [27,28]. Constraints on the size of these extra–dimensions can be obtained both from the laboratory physics [21] and from astrophysics and cosmology (see e.g. [29]). For instance, the emission of KK gravitons induces an energy loss in many astrophysical objects [30] such as the Sun, red giants and supernovae SN1987A [31] implying the lower bound  $M_D > 30 - 130 \text{ TeV}$  (2.1 - 9.2) TeV for the case of  $n = 2$  (3) extra–dimensions [30,31]. Some authors [20,25,32,33] also have recently investigated the possible presence of axions

---

\*Unité Mixte de Recherche du CNRS, UMR 8627.

in the bulk which would be coupled to the brane degrees of freedom. Such a bulk axion also gives rise to a tower of KK-states as seen from a four dimensional point of view.

The goal of this article is to investigate the effects of the photon–KK graviton and photon–KK axion oscillations and to estimate their effects in cosmology and astrophysics, as well as terrestrial experiments. Since there is a large number of KK states with which the photon can mix, one can expect a departure from the usual four dimensional result. We first describe (§ II) the photon–KK graviton system starting from a  $D$  dimensional action, linearising it and compactifying it to four dimensions. We also show (§ III) that the photon–axion mixing is described by the same formalism and lead to the same effects as for gravitons. Then, we turn to investigate the mixing in itself. For that purpose, we sum up the known results of the mixing of a photon with a low mass particle in four dimensions (§ IV A) and then discuss the most general case of a  $D$  dimensional spacetime (§ IV B). The general expression for the oscillation probability is then evaluated in the particular cases of a five (§ V) and of a six (§ VI) dimensional spacetime. We show that, as long as one is in a weak coupling regime, one can add the individual probabilities which leads to an enhancement of the oscillation probability if the KK modes are light enough. We also show that there exists a regime where the photon mixes strongly preferentially with a given KK mode. We generalise our results to an inhomogeneous magnetic field (§ VII) when the probability of oscillation is small, and then turn to the cosmological and astrophysical situations where such effects may be observed. We study the case of the cosmic microwave background (§ VIII A), of pulsars (§ VIII B) and of magnetars (§ VIII C). We show that even if the enhancement of the oscillation probability can be very important, it is still very difficult to observe this effect in known astrophysical systems. To finish (§ IX) we discuss laboratory experiments and particularly polarisation experiments. For that purpose, we describe the computation of the phase shift between the two polarisations of an electromagnetic wave and compare the result to the standard four dimensional case.

## II. EQUATIONS OF MOTION FOR THE PHOTON–GRAVITON SYSTEM

Following [27,28], we consider a field theory defined by the  $D$  dimensional action

$$S_D = -\frac{1}{2\kappa_D^2} \int d^D z \sqrt{-\bar{g}} \bar{R} + \int d^D z \sqrt{-\bar{g}} \mathcal{L}_m, \quad (2)$$

where  $\bar{g}_{AB}$  is the  $D$  dimensional metric with signature  $(-, +, \dots, +)$ ,  $\kappa_D^2 \equiv 8\pi G_D = \bar{M}_D^{-(2+n)}$  and  $\mathcal{L}_m$  is the matter Lagrangian. The indices  $A, B, \dots$  take the value  $0, \dots, 3, 5, \dots, D$  and we decompose  $z^A$  as

$$z^A = (x^\mu, y^a) \quad (3)$$

with  $\mu, \nu, \dots = 0, \dots, 3$  and  $a, b, \dots = 5, \dots, D$ .

This theory is considered as being a low energy effective theory valid below some cut-off  $M_{\max}$  in energy (see e.g. [20,21,25]). We will discuss in this paper only the cases  $n = 1$  and  $n = 2$  in details, and our conclusions, in these cases, are mostly cut-off independent. We stress that the relationship between this cut-off and the fundamental string scale (if one wishes to embed these theories in superstring models) can be much more complicated than what is naively expected (see [34]).

We expand the metric around the  $D$  dimensional Minkowski spacetime as

$$g_{AB} = \eta_{AB} + \frac{h_{AB}}{\bar{M}_D^{1+n/2}} \quad (4)$$

where  $\eta_{AB}$  is the  $D$  dimensional Minkowski metric. Inserting (4) in (2) and using the definition of the stress–energy tensor as  $\sqrt{-\bar{g}} T_{AB} \equiv 2\delta(\mathcal{L}_m \sqrt{-\bar{g}})/\delta g^{AB}$ , so that  $\mathcal{L}_m \sqrt{-\bar{g}} = \mathcal{L}_{m0} - h^{AB} T_{AB}/2\bar{M}_D^{1+n/2}$ , we obtain the linearised action

$$S_D = \int d^D z \left[ \frac{1}{2} h^{AB} \partial^C \partial_C h_{AB} - \frac{1}{2} h_A^A \partial^C \partial_C h_B^B + \frac{1}{2} h^{AB} \partial_A \partial_B h_C^C + \frac{1}{2} h_A^A \partial_C \partial_B h^{CB} - h^{AB} \partial_A \partial_C h_B^C - \frac{1}{\bar{M}_D^{1+n/2}} h^{AB} T_{AB} + \mathcal{L}_{m0} \right]. \quad (5)$$

We compactify this theory to get a four dimensional theory and use the periodicity on  $y^a$  to expand the field  $h^{AB}$  as

$$h_{AB}(z^A) = \sum_{\vec{p} \in \mathbb{Z}} \frac{h_{AB}^{(\vec{p})}(x^\mu)}{\sqrt{V_n}} \exp(i \frac{p^a y_a}{R}) \quad (6)$$

where  $V_n = (2\pi R)^n$  is the volume of the compact  $n$  dimensional space (assumed to be a cubic  $n$ -torus).  $h_{AB}$  is split into a sum of KK modes living in the four dimensional spacetime. The ordinary matter being confined to the brane, its stress-energy tensor must satisfy

$$T_{AB}(z^C) = T_{\mu\nu}(x^\lambda)\delta^{(n)}(y^c)\eta_A^\mu\eta_B^\nu. \quad (7)$$

In which follows, we restrict our attention to the case of an electromagnetic field  $F_{\mu\nu}$  for which the stress-energy tensor is given by

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu^\lambda - \frac{1}{4}\eta_{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}. \quad (8)$$

The fields  $h_{AB}^{(\vec{p})}$  can be decomposed into spin-2, spin-1 and spin-0 four dimensional fields [27,28]. Only spin-2 and spin-0 particles couple to ordinary matter and spin-0 particles couple only to  $T_\lambda^\lambda$ . For the electromagnetic field  $T_\lambda^\lambda = 0$  classically so that the only relevant KK modes (at the tree level analysis of this article) will be the spin-2 particles  $G_{\mu\nu}^{(\vec{p})}$  for which the action (5) reduces to [27,28]

$$S_4 = \int d^4x \left[ \frac{1}{2}G_{\mu\nu}^{(\vec{p})}(\square - m_{\vec{p}}^2)G^{(\vec{p})\mu\nu} + G^{(\vec{p})\mu\nu}\partial_\mu\partial_\lambda G_\nu^{(\vec{p})\lambda} - \frac{1}{2}G_\mu^{(\vec{p})\mu}(\square - m_{\vec{p}}^2)G_\nu^{(\vec{p})\nu} - G^{(\vec{p})\mu\nu}\partial_\mu\partial_\nu G_\lambda^{(\vec{p})\lambda} \right. \\ \left. - \frac{1}{M_4}G^{(\vec{p})\mu\nu}T_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right], \quad (9)$$

where  $m_{\vec{p}}^2 = \vec{p}^2/R^2$  is the square mass of the KK graviton,  $\square \equiv \partial_\mu\partial^\mu$ .

The equations of motion deduced from the Lagrangian (9) are the coupled Einstein-Maxwell equations

$$(\square - m_{\vec{p}}^2)G_{\mu\nu}^{(\vec{p})} = \frac{2}{M_4}T_{\mu\nu} \quad (10)$$

$$\partial^\mu G_{\mu\nu}^{(\vec{p})} = 0 \quad (11)$$

$$G_\mu^{(\vec{p})\mu} = 0 \quad (12)$$

$$\partial_\alpha F^{\alpha\beta} - \frac{2}{M_4} \sum_{\vec{p}} \partial_\alpha \left( G^{(\vec{p})\alpha\nu} F_\nu^\beta - G^{(\vec{p})\beta\nu} F_\nu^\alpha \right) = 0. \quad (13)$$

When  $\vec{p} \neq 0$ , the field  $G_{\mu\nu}^{(\vec{p})}$  has  $10-1-4=5$  degrees of freedom which is what is expected for a massive spin-2 particle.

We now consider an electromagnetic plane wave in the presence of a magnetic field  $\vec{H}_0$  which is assumed constant on a characteristic scale  $\Lambda_c$  in the sense that its variation in space and time are negligible on scales comparable to the photon wavelength and period. We define the basis

$$\vec{e}_\parallel \equiv \frac{\vec{k}}{k}, \quad \vec{e}_\times \equiv \frac{\vec{H}_{0\perp}}{H_{0\perp}}, \quad \vec{e}_+, \quad (14)$$

such that  $(\vec{e}_\times, \vec{e}_+, \vec{e}_\parallel)$  is a direct orthonormal basis of the three dimensional space and where  $\vec{H}_{0\perp}$  is the perpendicular component of  $\vec{H}_0$  with respect to the direction of propagation  $\vec{k}$ . We decompose the KK gravitons in scalar (S), vector (V) and tensor (T) as

$$(S) \quad G_{00}^{(\vec{p})} = \phi^{(\vec{p})}, \quad G_{0i}^{(\vec{p})} = -ik_i k^2 \dot{\phi}^{(\vec{p})}, \quad G_{ij}^{(\vec{p})} = \frac{\phi^{(\vec{p})}}{3}\delta_{ij} - \frac{3}{2k^2}\Delta_{ij} \left( \frac{\phi}{3} + \frac{\ddot{\phi}}{k^2} \right)^{(\vec{p})} \quad (15)$$

$$(V) \quad G_{0i}^{(\vec{p})} \equiv V_i^{(\vec{p})} = V_+^{(\vec{p})}e_i^+ + V_\times^{(\vec{p})}e_i^\times, \quad G_{00}^{(\vec{p})} = 0, \quad G_{ij}^{(\vec{p})} = \frac{2}{k^2}k_{(j}\dot{V}_{i)}^{(\vec{p})} \quad (16)$$

$$(T) \quad G_{00}^{(\vec{p})} = 0, \quad G_{0i}^{(\vec{p})} = 0, \quad G_{ij}^{(\vec{p})} = G_+^{(\vec{p})}\epsilon_{ij}^+ + G_\times^{(\vec{p})}\epsilon_{ij}^\times, \quad (17)$$

where  $\Delta_{ij} \equiv \left( k_i k_j - \frac{k^2}{3}\delta_{ij} \right)$ ,  $i, j = 1..3$  and a dot refers to a time derivative. The polarisation tensor of the graviton modes  $\epsilon_{ij}^\lambda$  is defined by

$$\epsilon_{ij}^\lambda \equiv (e_i^\times e_j^\times - e_i^+ e_j^+) \delta_\times^\lambda + 2e_i^{(+)} e_j^{(\times)} \delta_+^\lambda. \quad (18)$$

The advantage of such a decomposition is that the scalar, vector and tensor contributions decouple. The five degrees of freedom of each massive spin-2 KK gravitons have been decomposed in one scalar mode ( $\phi^{(\vec{p})}$ ), two vector modes ( $V_{+/\times}^{(\vec{p})}$ ) and two tensor modes ( $G_{+/\times}^{(\vec{p})}$ ). Each of these modes satisfies independently the constraints (11–12).

We consider an electromagnetic wave with a potential vector of the form

$$\vec{A} = i(A_{\times}(u), A_{+}(u), 0)e^{-i\omega t}, \quad (19)$$

where  $u$  is the coordinate along the direction of propagation. We have introduced the arbitrary phase  $i$  so that its electric and magnetic fields are

$$\vec{E} \equiv -\partial_t \vec{A} = (\omega A_{\times}(u), \omega A_{+}(u), 0)e^{-i\omega t} \quad (20)$$

$$\vec{B} \equiv \text{curl}(\vec{A}) = (-i\partial_u A_{+}(u), i\partial_u A_{\times}(u), 0)e^{-i\omega t}. \quad (21)$$

The stress-energy tensor of these waves in the presence of  $\vec{H}_0$  has no vector component. Its tensor component is given by

$$T_{ij} = i \sum_{\lambda=+,\times} \partial_u A_{\lambda} H_{0\perp} e^{-i\omega t} \epsilon_{ij}^{\lambda}. \quad (22)$$

We see, as expected, that a plane wave possesses a tensor part only if it propagates in an external field and that the polarisations  $+$  and  $\times$  of the electromagnetic wave couple respectively to the polarisations  $+$  and  $\times$  of the gravitons. The electromagnetic wave generates also scalar perturbations, but it can be shown [6] that (in the usual four dimensional case) the total energy converted in this scalar wave are negligible compared to the tensor contribution. In the following, we concentrate on the tensor modes.

The equation of evolution of this system is given by the Einstein equation (10) which reduces to

$$(\omega^2 + \partial_u^2 - m_{\vec{p}}^2) G_{\lambda}^{(\vec{p})} = \frac{2iH_{0\perp}}{M_4} \partial_u A_{\lambda}, \quad (23)$$

and the Maxwell equation (13) which reduces to

$$(\omega^2 + \partial_u^2) A_{\lambda} = \frac{2iH_{0\perp}}{M_4} \sum_{\vec{p}} \partial_u G_{\lambda}^{(\vec{p})}, \quad (24)$$

where we have used the ansatz  $G_{ij} = \sum_{\lambda} G_{\lambda}(u) e^{-i\omega t} \epsilon_{ij}^{\lambda}$  for the gravitons.

Since we have assumed that the magnetic field varies in space on scales much larger than the photon wavelength, we can perform the expansion  $\omega^2 + \partial_u^2 = (\omega + i\partial_u)(\omega - i\partial_u) = (\omega + k)(\omega - i\partial_u)$  for a field propagating in the  $+u$  direction. If we assume a general dispersion equation of the form  $\omega = nk$  and that the refractive index  $n$  satisfies  $|n - 1| \ll 1$ , we may approximate  $\omega + k = 2\omega$  and  $k/\omega = 1$ . This approximation can be understood as a WKB limit where we set  $A(u) = |A(u)|e^{iku}$  and assume that the amplitude  $|A|$  varies slowly, i.e. that  $\partial_u |A| \ll k|A|$ . In that limit, the system (23–24) reduces to

$$[\omega - i\partial_u + \mathcal{M}_{\lambda}] \begin{bmatrix} A_{\lambda} \\ G_{\lambda}^{(0)} \\ \vdots \\ G_{\lambda}^{(q)} \\ \vdots \end{bmatrix} = 0, \quad (25)$$

the matrix  $\mathcal{M}_{\lambda}$  being given by

$$\begin{pmatrix} \Delta_{\lambda} & \Delta_M & \Delta_M & \cdots & \Delta_M & \cdots & \cdots \\ \Delta_M & \Delta_m^{(0)} & 0 & \cdots & 0 & \cdots & \cdots \\ \Delta_M & 0 & \Delta_m^{(1)} & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \Delta_M & 0 & \cdots & 0 & \Delta_m^{(q)} & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad (26)$$

with

$$\Delta_M \equiv \frac{H_{0\perp}}{M_4} \quad \text{and} \quad \Delta_m^{(q)} \equiv \bar{p}_{(q)}^2 \Delta_m. \quad (27)$$

$\vec{p}_{(q)}$  is a n-uplets  $(p_{q_1}, p_{q_2}, \dots, p_{q_n})$  of integers and we have ordered the  $\Delta_m^{(q)}$  such that

$$\left| \Delta_m^{(q)} \right| \leq \left| \Delta_m^{(q+1)} \right|$$

and  $\Delta_m$  is defined by

$$\Delta_m \equiv \frac{-1}{2R^2\omega}. \quad (28)$$

Each  $\Delta_m^{(q)}$  appears with a multiplicity given by the number of n-uplets having the same norm  $\sum_{i=1\dots n} p_{q_i}^2$ . We define the two series  $(r_i)_{i \geq 1}$  and  $(s_i)_{i \geq 1}$  such that

$$\Delta_m^{(r_i-1)} < \Delta_m^{(r_i)} = \Delta_m^{(r_i+1)} = \dots \Delta_m^{(r_i+s_i-1)} < \Delta_m^{(r_i+s_i)} \equiv \Delta_m^{(r_{i+1})}. \quad (29)$$

We have  $r_{i+1} = r_i + s_i$ , and  $s_i$  is the multiplicity of the element  $\Delta_m^{(r_i)}$ , i.e. the number of times it appears in the matrix (26).  $r_i$  is the rank in the series  $(\Delta_m^{(0)}, \Delta_m^{(1)}, \dots)$  where the  $i^{\text{th}}$  distinct value of  $\Delta_m^{(q)}$  appears for the first time. In the case of a five dimensional spacetime one can easily find out that  $s_1 = 1$  and  $s_i = 2$  for  $i > 1$  and that  $r_1 = 0, r_2 = 1, r_3 = 3, \dots$ . In the case of a six dimensional spacetime,  $s_i = (1, 4, 4, \dots)$ ,  $r_i = (0, 1, 5, 9, \dots)$ . Introducing the cut-off  $M_{\text{max}}$  discussed above, we require  $m_{\vec{p}}^2 = \bar{p}^2/R^2 < M_{\text{max}}^2$ , which using (1) translates into  $\bar{p}^2 < p_{\text{max}}^2$  with

$$p_{\text{max}} = \left( \frac{\bar{M}_4}{M_D} \right)^{2/n} \left( \frac{M_{\text{max}}}{M_D} \right). \quad (30)$$

Setting  $M_{\text{max}} \sim M_D$ , we obtain  $p_{\text{max}} \sim (\bar{M}_4/M_D)^{2/n}$ . For  $n = 2$  and  $M_D \sim 1\text{TeV}$  one finds

$$p_{\text{max}} \sim 10^{15}, \quad (31)$$

which means that we have to consider a very large number of KK states. We also define a maximum index,  $N$  say, for the series  $\Delta_m^{(q)}$  defined by

$$N \equiv \sup\{q \mid \bar{p}_{(q)}^2 = p_{\text{max}}^2\}, \quad (32)$$

which translates into a maximum index  $N_D$  for the series  $s_i$  and  $r_i$ . We stress that the number of KK modes relevant for the photon–KK graviton oscillation is likely to be smaller than  $N$  due to decoherence effects, such as the source and detector finite width in momentum, the wave packet separation for massive (and non-relativistic) KK modes... (see e.g. [35] for a description of these effects in the case of neutrino oscillation).

The term  $\Delta_\lambda$  can be decomposed as  $\Delta_\lambda = \Delta_{\text{QED}} + \Delta_{\text{CM}} + \Delta_{\text{plasma}}$ . The first term contains the effect of vacuum polarisation giving a refractive index to the photon (see e.g. Adler [3]) and can be computed by adding the Euler–Heisenberg effective Lagrangian which is the lowest order term of the non-linearity of the Maxwell equations in vacuum (see e.g. [36,37]) to the action (9)<sup>1</sup>. The second term describes the Cotton–Mouton effect, i.e. the birefringence of gases and liquids in presence of a magnetic field and the third term the effect of the plasma (since, in general, the photon does not propagate in vacuum). Their explicit expressions are given by

$$\begin{aligned} \Delta_{\text{QED}}^\times &= \frac{7}{2}\omega\xi, & \Delta_{\text{QED}}^+ &= 2\omega\xi, \\ \Delta_{\text{plasma}} &= -\frac{\omega_{\text{plasma}}^2}{2\omega}, \\ \Delta_{\text{CM}}^\times - \Delta_{\text{CM}}^+ &= 2\pi CH_0^2, \end{aligned} \quad (33)$$

---

<sup>1</sup>The equation of motion derived from (9) is (25) with  $\Delta_\lambda = 0$ . We intentionally omit the Euler–Heisenberg contribution in the presentation for the sake of clarity. Its Lagrangian is explicitly given by  $\mathcal{L}_{EH} = \frac{\alpha^2}{90m_e^4} [(F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}(F^{\mu\nu}\tilde{F}_{\mu\nu})^2]$ .

with  $\xi \equiv (\alpha/45\pi)(H_{0\perp}/H_c)^2$ ,  $H_c \equiv m_e^2/e = 4.41 \times 10^{13}$  G,  $m_e$  the electron mass,  $e$  the electron charge and  $\alpha$  the fine structure constant.  $C$  is the Cotton–Mouton constant [38]; this effect gives only the difference of the refractive indices and the exact value of  $C$  is hard to determine [39]; we will neglect this effect but for the polarisation experiments (see § IX B). The plasma frequency  $\omega_{\text{plasma}}$  is defined by

$$\omega_{\text{plasma}}^2 \equiv 4\pi\alpha \frac{n_e}{m_e}, \quad (34)$$

$n_e$  being the electron density. Note that the  $\Delta_m^{(q)}$  are always negative whereas  $\Delta_\lambda$  is positive if the contribution of the vacuum dominates and negative when the plasma term dominates.

The equation of motion (25) reduces to the one studied by Raffelt and Stodolsky [4] when one considers only four dimensions so that  $\mathcal{M}_\lambda$  contains only the massless graviton and is then a  $2 \times 2$  matrix. The main difference lies in the fact that now the electromagnetic component couples to a large numbers of KK gravitons. This can be compared to some models of neutrino oscillations in spacetime with extra–dimensions [40,41]. We should also note that the two polarisations are, as expected, completely decoupled and obey the same equation of evolution. In the following of this article, we omit the subscript  $\lambda$  of the polarisation.

### III. EQUATIONS OF MOTION FOR THE PHOTON–AXION SYSTEM

Before turning to the study the photon–graviton mixing, we consider the case of axions and show that the photon–axion mixing can be described by the same formalism.

We consider the generic action [32,33] for the bulk axion photon system

$$S_4 = \int d^4x \left[ \sum_{\vec{p}} \left( -\frac{1}{2} \left\{ \partial^\mu a^{(\vec{p})} \partial_\mu a^{(\vec{p})} + m_{\vec{p}}^2 a^{(\vec{p})2} \right\} + \frac{a^{(\vec{p})}}{f_{\text{PQ}}} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (35)$$

where the  $a^{(\vec{p})}$  are the mass eigenstates of the axions and  $m_{\vec{p}}$  their masses.  $\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$  is the dual of the electromagnetic tensor,  $\epsilon_{\mu\nu\rho\sigma}$  being the completely antisymmetric tensor such that  $\epsilon_{0123} = +1$ . As for the gravitons, the mass spectrum is expected to be discrete, the states can be labelled by a n-uplet  $\vec{p}$  and is expected to have a typical spacing of  $1/R$ . We have considered here that every axion KK state  $a^{(\vec{p})}$  couples to the photon with the same coupling  $1/f_{\text{PQ}}$ . This is only expected to be true if the typical mass,  $m_{\text{PQ}}$ , given to the axion zero mode by instanton effects is much lower than the typical KK mass  $1/R$  [33]. Let us further stress here that for such bulk axions the usual relationship between the axion mass and the PQ scale does not hold anymore, so that one expects to see interesting new effects to appear [33]. Inspired by the usual bounds on  $f_{\text{PQ}}$ , we take  $f_{\text{PQ}}$  of order  $10^{10}$  GeV. However we stress that the usual bounds on  $f_{\text{PQ}}$  may be modified partly because of a large number of axion–like particles coupling to the standard model fields. For example, we expect that the astrophysical bounds will be more stringent mainly because a star will now be able to emit all the energetically accessible modes (see [32] and also [33] for a discussion on relic axions oscillations).

We do not consider the perturbations of the metric and work in Minkowski spacetime since we are interested in the interaction between the photon and the axion. We deduce from (35) the coupled Klein–Gordon and Maxwell equations

$$(\square - m_{\vec{p}}^2) a^{(\vec{p})} = -\frac{1}{f_{\text{PQ}}} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (36)$$

$$\partial_\alpha F^{\alpha\beta} = \frac{4}{f_{\text{PQ}}} \partial_\alpha \left[ \sum_{\vec{p}} a^{(\vec{p})} \tilde{F}^{\alpha\beta} \right]. \quad (37)$$

We now decompose the electromagnetic wave as in (19–21) with respect to the basis (14), so that the former system reads

$$(\square - m_{\vec{p}}^2) a^{(\vec{p})} = \frac{4H_{0\perp}}{f_{\text{PQ}}} A_\times \quad (38)$$

$$\square A_\lambda = \frac{4H_{0\perp}}{f_{\text{PQ}}} \omega \delta_{\lambda\times} \sum_{\vec{p}} a^{(\vec{p})}, \quad (39)$$

where we have decomposed the axions as  $a^{(\vec{p})}(u) \exp(-i\omega t)$ . Using the same WKB limit as in section IV, we obtain the linearised system

$$(\omega - i\partial_u + \mathcal{M}) \begin{bmatrix} A_+ \\ A_\times \\ a^{(0)} \\ \vdots \\ a^{(\vec{p})} \\ \vdots \end{bmatrix} = 0. \quad (40)$$

The matrix  $\mathcal{M}$  is now defined by

$$\mathcal{M} = \begin{pmatrix} \Delta_+ & 0 & 0 & 0 & 0 & \cdots \\ 0 & \Delta_\times & \Delta_M & \Delta_M & \cdots & \\ 0 & \Delta_M & \Delta_a^{(0)} & 0 & 0 & \cdots \\ 0 & \Delta_M & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \Delta_a^{(q)} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad (41)$$

with

$$\Delta_M = \frac{4H_{0\perp}}{f_{\text{PQ}}}, \quad \Delta_a^{(\vec{p})} = -\frac{m_p^2}{2\omega}, \quad (42)$$

$\Delta_+$  and  $\Delta_\times$  being given by equation (33). This system reduces to the Raffelt and Stodolsky [4] system when we consider only four dimensions. Only the component  $\times$ , i.e. parallel to the magnetic field, couples to the axions. This is a major difference compared with gravitons for which both polarisations of the photon evolve alike whereas here only  $A_\times$  is affected by the mixing.

One of the goal of this section was to set the theoretical framework for further experimental studies of photon–bulk axion oscillations (see § IX for a more detailed discussion) and to show that it is described by a similar formalism as photon–KK graviton oscillations (under the validity conditions explained below equation (35)).

#### IV. PHOTON–KK STATE MIXING IN A HOMOGENEOUS FIELD

We now describe the physical implications of the system (23–24) and start by reviewing briefly the well studied problem of the mixing of a photon with a low mass particle in four dimensions (§ IV A). We then give the exact expression of the oscillation probability in  $D$  dimensions and discuss qualitatively its magnitude and the effect of the coupling of the photon to a large number of particles.

##### A. The usual photon mixing with a low mass particle

This case was well studied in the literature, see e.g. Raffelt and Stodolsky [4] and we just summarize the main features of the results to compare to the case of a spacetime with extra–dimensions. For the mixing with a single particle of mass  $m$ , the matrix  $\mathcal{M}$  reduces to

$$\mathcal{M} = \begin{pmatrix} \Delta_\lambda & \Delta_M \\ \Delta_M & \Delta_m \end{pmatrix} \quad (43)$$

with  $\Delta_m \equiv -m^2/2\omega$ . The solution to the equation of motion (25) is obtained by diagonalising  $\mathcal{M}$  through a rotation

$$\begin{bmatrix} A' \\ G' \end{bmatrix} = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \begin{bmatrix} A \\ G \end{bmatrix} \quad (44)$$

with

$$\tan 2\vartheta \equiv 2 \frac{\Delta_M}{\Delta_\lambda - \Delta_m} = \frac{2\alpha}{1-\beta} \quad (45)$$

and where we have introduced  $\alpha \equiv \Delta_M/\Delta_\lambda$  and  $\beta \equiv \Delta_m/\Delta_\lambda$ . We obtain by solving (25) in this new basis

$$\begin{aligned} A'(u) &= e^{-i\Delta'_\lambda u} A'(0) \\ G'(u) &= e^{-i\Delta'_g u} G'(0) \end{aligned} \quad (46)$$

where a global phase  $\omega u$  has been omitted. The two eigenvalues  $\Delta'_\lambda$  and  $\Delta'_g$  of  $\mathcal{M}$  are explicitly given by

$$\Delta'_\lambda = \frac{\Delta_\lambda + \Delta_m}{2} + \frac{\Delta_\lambda - \Delta_m}{2 \cos 2\vartheta} \quad \text{and} \quad \Delta'_g = \frac{\Delta_\lambda + \Delta_m}{2} - \frac{\Delta_\lambda - \Delta_m}{2 \cos 2\vartheta}. \quad (47)$$

Going back to the initial basis, we obtain

$$\begin{aligned} A(u) &= \left( e^{-i\Delta'_\lambda u} \cos^2 \vartheta + e^{-i\Delta'_g u} \sin^2 \vartheta \right) A(0) + \sin \vartheta \cos \vartheta \left( e^{-i\Delta'_\lambda u} - e^{-i\Delta'_g u} \right) G(0), \\ G(u) &= \sin \vartheta \cos \vartheta \left( e^{-i\Delta'_\lambda u} - e^{-i\Delta'_g u} \right) A(0) + \left( e^{-i\Delta'_g u} \cos^2 \vartheta + e^{-i\Delta'_\lambda u} \sin^2 \vartheta \right) G(0). \end{aligned} \quad (48)$$

The oscillation probability of a photon into a graviton is computed by considering the initial state ( $A(0) = 1, G(0) = 0$ ) and is given by

$$P(\gamma \rightarrow g) \equiv |\langle A(0) | G(u) \rangle|^2 = \sin^2(2\vartheta) \sin^2 \left( \frac{\Delta_{\text{osc}}}{2} u \right), \quad (49)$$

$$= (\Delta_M u)^2 \frac{\sin^2(\Delta_{\text{osc}} u/2)}{(\Delta_{\text{osc}} u/2)^2} \quad (50)$$

with

$$\Delta_{\text{osc}} \equiv \Delta'_\lambda - \Delta'_g = \frac{\Delta_\lambda - \Delta_m}{\cos 2\vartheta} = \frac{2\Delta_M}{\sin 2\vartheta} = \frac{1-\beta}{\cos 2\vartheta} \Delta_\lambda \quad (51)$$

The oscillation length is thus given by  $\ell_{\text{osc}} \equiv 2\pi/\Delta_{\text{osc}}$ . We see that the oscillation probability cannot be larger than  $(\Delta_M u)^2$ . One has to be aware that  $\vartheta$  depends on the polarisation index  $\lambda$ .

It is interesting to single out the two following limiting regimes:

- The *weak mixing* regime in which  $\vartheta \ll 1$  so that the probability (49) reduces to

$$P(\gamma \rightarrow g) = 4 \frac{\alpha^2}{(1-\beta)^2} \sin^2 \left( \frac{1-\beta}{2} \Delta_\lambda u \right). \quad (52)$$

When the oscillation length  $\ell_{\text{osc}} = \frac{2\pi\vartheta}{\Delta_M}$  is large with respect to the coherent path distance  $u$ , the weak mixing probability can be further approximated (with  $\Delta_M u \ll \vartheta \ll 1$ ) by

$$P(\gamma \rightarrow g) \simeq (\alpha \Delta_\lambda u)^2 \simeq (\Delta_M u)^2. \quad (53)$$

- The *strong mixing* regime in which the mixing is maximal, i.e. when  $\vartheta \simeq \pi/4$ , so that the oscillation probability reduces to

$$P(\gamma \rightarrow g) = \sin^2(\Delta_M u) \quad (54)$$

and the oscillation length to

$$\ell_{\text{osc}} = \frac{\pi}{\Delta_M}. \quad (55)$$

A complete transition between a photon and the light particle is then possible. This can only happen when  $\Delta_m$  and  $\Delta_\lambda$  have the same sign (see equation (45)). We further note here that the width in  $\beta$  of the strong mixing region is roughly given by  $\alpha$  according to equation (45).



## B. Mixing in $D$ dimensions

### 1. General result

To compute the oscillation probability in a spacetime with extra-dimensions, we first have to solve (25) which implies the diagonalisation of the matrix (26). We present the explicit and detailed computation of both the eigenvalues and eigenvectors in appendix A. We then compute in appendix B the explicit form of the oscillation probability (see equation (B5))

$$P(\gamma \rightarrow g) = 1 - \left| \sum_{i=1}^{N_D} f_{x_i}^2 e^{ix_i u} \right|^2. \quad (56)$$

Taking into account the fact that  $\sum f_{x_i}^2 = 1$ , it can be rewritten as

$$P(\gamma \rightarrow g) = 2 \sum_{i,j=1}^{N_D} f_{x_i}^2 f_{x_j}^2 \sin^2 \left[ \frac{x_i - x_j}{2} u \right], \quad (57)$$

where the coefficients  $f_{x_j}^2$  are defined by (see equation (B6))

$$f_{x_j}^2 \equiv \left[ 1 + \alpha^2 \sum_{i=1}^{N_D} \frac{s_i}{(y_j - \beta_i)^2} \right]^{-1}. \quad (58)$$

The expressions (57) and (58) depend on the eigenvalues  $y_i$  solutions of the equation (A9). Introducing the notations  $y \equiv x/\Delta_\lambda$ ,  $\alpha \equiv \Delta_M/\Delta_\lambda$ ,  $\beta \equiv \Delta_m/\Delta_\lambda$  and  $\beta_i \equiv \Delta_m^{(r_i)}/\Delta_\lambda$ , the eigenvalues equation (A9) can be rewritten as

$$y - 1 = \alpha^2 \sum_{i=1}^{N_D} \frac{s_i}{y - \beta_i}. \quad (59)$$

The photon-KK graviton oscillations is then completely described by the set of equations (57–59).

Indeed, it is difficult (even impossible if  $n > 1$ ) to compute analytically the roots of (59). For instance the coefficients  $s_i$  are not known analytically if  $n > 1$ ; it is of course possible to compute  $P(\gamma \rightarrow g)$  numerically, but this is not our purpose here. In the next two sections we derive the oscillation probability in the two cases  $n = 1$  and  $n = 2$  in a range of parameters dictated by the systems where such a mixing may appear. In the next paragraph, we discuss qualitatively the results found there, stressing some new effects due to the presence of a large number of KK states, as well as to the degeneracy of each KK level.

### 2. Qualitative discussion

We only discuss the cases where  $\alpha$  is small in comparison to  $\beta$  as dictated by the physical systems studied in § VIII and § IX. Two different limiting regimes appear, a *large radius* regime (when  $|\beta|$  is smaller than unity) and for which there is a significant effect of the extra-dimensions, and a *small radius* regime (when  $|\beta|$  is larger than unity) and for which there is only small departure from the usual four dimensional photon mixing.

Let us first discuss the large radius regime where  $|\beta|$  is smaller than unity. As in four dimensions, according to the respective value of  $\Delta_\lambda$  and of the  $\Delta_m^{(q)}$ 's, two behaviours can appear:

- A *strong mixing* regime with one given KK state, if there exists a state  $K$  such that

$$|\Delta_m^{(K)} - \Delta_\lambda| \ll \sqrt{s_K} \Delta_M. \quad (60)$$

This can happen only if  $\beta > 0$ , i.e. when plasma effects dominate over the vacuum polarisation. We stress also that since we have assumed along this discussion that  $\alpha$  is lower than  $\beta$  there is at most one KK state which can mix strongly with the photon. The total probability will be found to be dominated by a term of the form

$$P(\gamma \rightarrow g) = (1 - \eta) \sin^2 \left[ \Delta_{\text{osc}}^{(K)} u \right] + 4 \sum_{i \neq K} \frac{s_i \alpha^2}{(1 - \beta_i)^2} \sin^2 \left( \frac{1 - \beta_i}{2} \Delta_\lambda u \right), \quad (61)$$

with

$$\Delta_{\text{osc}}^{(K)} = \sqrt{s_K} \Delta_M \quad (62)$$

(see § V and § VI for a detailed derivation).  $\eta$  is much smaller than unity. As will be shown later, this form accounts for keeping only the dominant part of each  $f_{x_i}^2$ . Depending on the argument of the sines, the probability is either dominated by  $\sin^2 \left[ \Delta_{\text{osc}}^{(K)} u \right]$  or by the correction coming from the modes  $i \neq K$ . This shows a first departure to the four dimensional case due to the degeneracy of the KK level  $K$ ; the oscillation length associated with the strong mixing state (labelled by  $K$ ) is lowered by a factor  $\sqrt{s_K}$  which can be very large. Moreover, the width of the region in  $\Delta_\lambda$  of strong mixing is larger by a factor  $\sqrt{s_K}$  than in the usual case [see below equation (55)].

An other important difference with the usual four dimensional situation, where the strong mixing regime can only occur when  $\Delta_\lambda$  crosses *the unique*  $\Delta_m$  characteristic of the mixing state, we now have more possibilities to be in that regime, where a complete transition between the photon and the graviton is possible. Because of the presence of a KK state  $\Delta_m^{(q)}$  in any interval in  $\Delta_\lambda$  of typical width  $\Delta_m$ , only *fluctuations* of  $\Delta_\lambda$  of order  $\Delta_m$  can lead to it.

- A *weak mixing* regime where for all  $q$ ,  $|\beta_q - 1| \gg \alpha$ . The oscillation probability is then given by

$$P(\gamma \rightarrow g) \simeq 4 \sum_i \frac{s_i \alpha^2}{(1 - \beta_i)^2} \sin^2 \left( \frac{1 - \beta_i}{2} \Delta_\lambda u \right). \quad (63)$$

This contribution is exactly the one that will be intuitively thought of and obtained by summing the individual oscillation probabilities (52) of the photon into each KK state with the mixing angle

$$\tan 2\vartheta_q \equiv \frac{2\alpha}{1 - \beta_q}. \quad (64)$$

There are roughly three contributions to the sum (63) that we estimate as follows.

1. All the states such that  $|\beta_q| \ll 1$  mix with the photon with approximatively the same angle  $\vartheta_q \sim \alpha$  if we neglect  $\beta_q$  with respect to unity in (64). The order of magnitude of the probability of oscillations with these states is then

$$P(\gamma \rightarrow g) \sim 4\mathcal{N}_1 \alpha^2 \sin^2 \left( \frac{\Delta_\lambda}{2} u \right). \quad (65)$$

$\mathcal{N}_1$  can be estimated by counting the number of modes such that  $\beta_q \leq \beta_{N_1} \simeq 1$  with their multiplicity (see appendix C), i.e.  $\mathcal{N}_1 \sim \sum_{k=1}^{N_1} k^{n-1} \sim N_1^n \sim \beta^{-n/2}$  so that

$$P(\gamma \rightarrow g) \sim \frac{\alpha^2}{\beta^{n/2}} \sin^2 \left( \frac{\Delta_\lambda}{2} u \right). \quad (66)$$

This already shows that the oscillation probability can be greatly enhanced (by a factor  $\beta^{-n/2}$  with respect to the four dimensional case with the same mixing parameters [obtained from equation (52) with  $|\beta| \ll 1$ ]).

2. All  $\beta_q$  such that  $|\beta_q| \gg 1$  have a mixing angle roughly estimated by  $\vartheta_q \sim \alpha/\beta_q$  and their contribution to the probability is of order

$$P(\gamma \rightarrow g) \sim 4\alpha^2 \sum_{\beta_q > 1} \frac{1}{\beta_q^2} \sin^2 \left( \frac{\Delta_\lambda \beta_q}{2} u \right). \quad (67)$$

This series is difficult to evaluate since the oscillation length is different for each KK state. When  $n \leq 3$  it can be bounded by  $\alpha^2/\beta^{n/2}$  so that this contribution is at most of the same order of magnitude than the previous one.

3. The contribution of the  $\beta_q$  such that  $\beta_q \sim 1$  which only exists if  $\beta > 0$  is bounded by  $\alpha^2 \sum_{\beta_i \sim 1} s_i / (\beta_i - 1)^2$ . First of all, since  $|\beta_q - 1| \gg \alpha^2$  we are never in a strong mixing regime. Now, we single out  $\beta_K$ , the closest  $\beta_i$  to unity from which it follows that  $\forall i \neq K, |\beta_i - 1| \geq \beta/2$  and thus the contribution of all the  $\beta_i \sim 1$  for  $i \neq K$  is bounded by  $(\alpha^2/\beta^2) \sum_{\beta_i \sim 1, i \neq K} s_i$ . It can be dominated by the contribution of the term  $K$  given by  $\alpha^2 s_K / |\beta_K - 1|^2$  according to the relative value of  $|\beta_K - 1|$  in units of  $\beta$ .

In conclusion, the weak mixing case is characterised by an enhancement of the probability by a factor at least  $\beta^{-n/2}$  due to the fact that the photon couples to a large number of KK states. We further note that when  $\beta < 0$  and  $|\beta| \ll 1$  one can obtain an absolute bound on the oscillation probability of order  $\alpha^2/\beta^2$  for  $n \leq 3$  (when  $n = 2$ , this bound is given by  $10Q\alpha^2/\beta^2$  [see appendix (C)]).

We now turn to the *small radius* regime where  $|\beta| \gg 1$  and in which the photon mixes preferentially with the zero mode. The probability (57) can be expressed as

$$P(\gamma \rightarrow g) = (1 - \epsilon)P_{4D}(\gamma \rightarrow g) + 4 \sum_{i>1} \frac{\alpha^2}{(1 - \beta_i)^2} \sin^2 \left( \frac{\Delta_\lambda(1 - \beta_i)}{2} u \right), \quad (68)$$

where  $P_{4D}$  is the oscillation probability for the mixing with the zero mode and is given by (49) and  $\epsilon \ll 1$  is the correction of the oscillation probability with this mode coming from the existence of the extra-dimensions. In this case, the lightest massive KK mode is so heavy compared to the photon effective mass that it can barely be excited by the photon. The contribution of the other KK modes can be shown to be bounded by  $\mathcal{O}(\alpha^2\beta^{-2})$ . The contribution of the massive KK states is *suppressed* by a factor  $\beta^2 \gg 1$ .

Introducing the Compton wavelength,  $\lambda_\gamma$  say, associated with the effective mass of the photon and defined by

$$\lambda_\gamma \equiv |\omega \Delta_\lambda|^{-1/2}, \quad (69)$$

the required condition to be in a large radius regime can be rephrased as  $\lambda_\gamma < R$ , i.e. that the average scale associated with the photon is smaller than the radius of the extra-dimensions. The latter is expected to be of the order of the centimeter for two extra-dimensions.

In the two following sections, we derive these results in details for a five and six dimensional spacetime. Let us stress here that when  $n > 3$  the oscillation probability will strongly depend on the cut-off in energy in which case a more precise knowledge of the whole theory and the exact experimental situation (to take into account decoherence effect) are needed. We emphasize that in the following we have set the cut-off to its maximum value in order to be very general. The computed probability is thus the maximum one and the bounds on the parameters used to derive it are the most stringent.

## V. ESTIMATION OF THE PROBABILITY IN A FIVE DIMENSIONAL SPACETIME

In this section, we present the computation of the eigenvalues and of the oscillation probability when  $n = 1$ . As it will be seen the computation is easier in this case because the sums in (58-59) can be expressed in terms of circular or hyperbolic functions. Although the case  $n = 1$  is generally regarded as in contradiction with observation (see however e.g. [42]), this computation will teach us a lot about the mixing with a large number of particles.

In a five dimensional world, the mixing matrix  $\mathcal{M}$  is explicitly given by

$$\mathcal{M}_\lambda = \begin{pmatrix} \Delta_\lambda & \Delta_M & \Delta_M & \Delta_M & \Delta_M & \Delta_M & \cdots \\ \Delta_M & 0 & 0 & 0 & 0 & 0 & \cdots \\ \Delta_M & 0 & \frac{-1}{2R^2\omega} & 0 & 0 & 0 & \cdots \\ \Delta_M & 0 & 0 & \frac{-1}{2R^2\omega} & 0 & 0 & \cdots \\ \Delta_M & 0 & 0 & 0 & \frac{-4}{2R^2\omega} & 0 & \cdots \\ \Delta_M & 0 & 0 & 0 & 0 & \frac{-4}{2R^2\omega} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \quad (70)$$

This matrix can be compared to the one obtained for neutrino oscillations in spacetime with extra-dimensions (see e.g. [41]). We see on that example that for  $q \neq 0$  each  $\Delta_m^{(q)}$  is twice degenerated so that  $r_1 = 1$ ,  $r_i = 2i - 1$  and  $s_i = 2$  for  $i > 1$ .

The characteristic eigenvalues equation  $\det(\mathcal{M} - xI) = 0$ , in that simple case, can be obtained by developing the determinant of order  $2N + 2$  with respect to the last line to get a recursion relation with the determinant of order  $2N$  and to find the limit of this series. Indeed it leads to the same result that the general equation (A5).

With the notations of the former paragraph, the eigenvalues equation (A5) can now be rewritten after resummation (see 1.217 in [43]) as

$$y - 1 = \frac{\alpha^2}{\beta} \mathcal{K}(y/\beta), \quad (71)$$

where the function  $\mathcal{K}$  is defined by

$$\mathcal{K}(x) \equiv \pi|x|^{-1/2} \begin{cases} \cot \pi|x|^{1/2} & (x > 0) \\ -\coth \pi|x|^{1/2} & (x < 0). \end{cases} \quad (72)$$

The oscillation probability is then given by (57) where the coefficients (58) are now reexpressed after resummation as

$$f_{x_j}^2 = \left[ 1 + \frac{\alpha^2}{\beta^2} \mathcal{I}(y_j/\beta) \right]^{-1} \quad (73)$$

where the function  $\mathcal{I} \equiv \sum_{k \in \mathbb{Z}} (x - k^2)^{-2}$  is obtained from (72) as

$$\mathcal{I}(x) \equiv \frac{\pi}{2|x|^{3/2}} \begin{cases} \cot \pi\sqrt{x} + \pi\sqrt{x}(1 + \cot^2 \pi\sqrt{x}) & (x > 0) \\ \coth \pi\sqrt{|x|} - \pi\sqrt{|x|}(1 - \coth^2 \pi\sqrt{|x|}) & (x < 0). \end{cases} \quad (74)$$

We now determine an approximation of the solutions of the eigenvalues equation (71) and then of the oscillation probability in the cases  $\beta < 0$  and  $\beta > 0$  assuming that  $\alpha^2 \ll 1$ .

**A.  $\beta < 0$**

### 1. Eigenvalues

Setting  $\bar{\beta} \equiv -\beta > 0$ , we have  $\beta_j = -(j-1)^2 \bar{\beta}$  and one can easily sort out that the solutions of equation (71) are such that

$$y_1 > 1, \quad y_{j+1} \in ]\beta_{j+1}, \beta_j[ \quad (75)$$

with

$$y_1 \quad \text{solution of} \quad y - 1 = \pi \frac{\alpha^2}{\bar{\beta}} \sqrt{\frac{\bar{\beta}}{y}} \coth \pi \sqrt{\frac{y}{\bar{\beta}}}, \quad (76)$$

$$y_{j>1} \quad \text{solutions of} \quad 1 - y = \pi \frac{\alpha^2}{\bar{\beta}} \sqrt{\frac{\bar{\beta}}{-y}} \cot \pi \sqrt{\frac{-y}{\bar{\beta}}}. \quad (77)$$

In figure 1, we depict the graphical resolution of this equation.

The resolution of equation (71) then splits into the three following cases:

1.  $y_1$ : When  $y_1/\bar{\beta} \ll 1$ , (76) reduces to  $y - 1 = \alpha^2/y$  so that  $y_1 \simeq 1 + \alpha^2$ , and one can then check that the condition  $y_1/\bar{\beta} \ll 1$  is equivalent to  $\bar{\beta} \gg 1$ .

When  $y_1/\bar{\beta} \gg 1$ , we set  $y_1 = 1 + \epsilon$  with  $\epsilon > 0$  and (76) implies that  $0 < \epsilon = \pi \alpha^2 (1 + \epsilon)^{-1/2} / \bar{\beta}^{1/2} < \pi \alpha^2 / \bar{\beta}^{1/2}$ . Then if  $\alpha^2 / \bar{\beta}^{1/2} \ll 1$  we deduce that  $y_1 \simeq 1 + \pi \alpha^2 / \bar{\beta}^{1/2}$  and that the initial condition on  $y_1$  is equivalent to  $\bar{\beta} \ll 1$ .

In conclusion, if  $\alpha^2 / \bar{\beta}^{1/2} \ll 1$ ,

$$y_1 \simeq \begin{cases} 1 + \alpha^2 & (\bar{\beta} \gg 1) \\ 1 + \pi \frac{\alpha^2}{\bar{\beta}^{1/2}} & (\bar{\beta} \ll 1). \end{cases} \quad (78)$$

Note that these two solutions can be rewritten under the more compact form

$$y_1 \simeq 1 + \frac{\alpha^2}{\beta} \mathcal{K}(\beta^{-1}). \quad (79)$$

2.  $y_{j+1}; j > 1$ : Since  $y_{j+1}$  satisfies

$$(j-1) < \sqrt{\frac{-y_{j+1}}{\bar{\beta}}} < j$$

we set  $\sqrt{-y_{j+1}/\bar{\beta}} \equiv (j-1) + \epsilon_{j+1}$  with  $0 < \epsilon_{j+1} < 1$  and equation (77) rewrites as

$$1 + \bar{\beta}(j-1)^2 \left[1 + \frac{\epsilon_{j+1}}{j-1}\right]^2 = \frac{\pi\alpha^2}{\bar{\beta}(j-1)} \frac{\cot \pi\epsilon_{j+1}}{1 + \frac{\epsilon_{j+1}}{j-1}}. \quad (80)$$

Now, if  $\alpha^2/\bar{\beta} \ll 1$ , the l.h.s. of (80) being larger than unity, it implies that  $\cot \pi\epsilon_{j+1} \gg 1$  which thus behaves as  $1/\pi\epsilon_{j+1}$ . We can then solve (80) for  $\epsilon_{j+1}$  to get

$$y_j \simeq \beta_{j-1} - \frac{2\alpha^2}{1 - \beta_{j-1}}. \quad (81)$$

This expansion is valid whatever the magnitude of  $\bar{\beta}$  as long as  $\alpha^2/\bar{\beta} \ll 1$ .

3.  $y_2$ :  $y_2$  is the solution of (77) such that  $0 < \sqrt{-y_2/\bar{\beta}} < 1/2$ . Setting  $z \equiv -y_2/\bar{\beta}$ , (77) leads to

$$1 + \bar{\beta}z = \pi \frac{\alpha^2}{\bar{\beta}} \frac{\cot \pi\sqrt{z}}{\sqrt{z}} \quad (82)$$

with  $0 < \sqrt{z} < 1/2$ . The l.h.s. of (82) being greater than unity, it implies that, when  $\alpha^2/\bar{\beta} \ll 1$ ,  $\cot(\pi\sqrt{z})/\sqrt{z} \gg 1$  and thus behaves as  $1/\pi\sqrt{z}$ . At lowest order (82) then leads to  $z \simeq \alpha^2/\bar{\beta}$  and then

$$y_2 \simeq -\alpha^2. \quad (83)$$

Again, this solution is valid whatever  $\bar{\beta}$  such that  $\alpha^2/\bar{\beta} \ll 1$ .

4. Summary: When  $\alpha^2 \ll 1$ , the roots of (71) are well approximated by

$$\begin{aligned} y_1 &\simeq 1 + \frac{\alpha^2}{\beta} \mathcal{K}(\beta^{-1}) \\ y_{j>1} &\simeq \beta_{j-1} - \frac{s_{j-1}\alpha^2}{1 - \beta_{j-1}} \end{aligned} \quad (84)$$

for all  $\beta$  such that  $\alpha^2 \ll |\beta|$ .

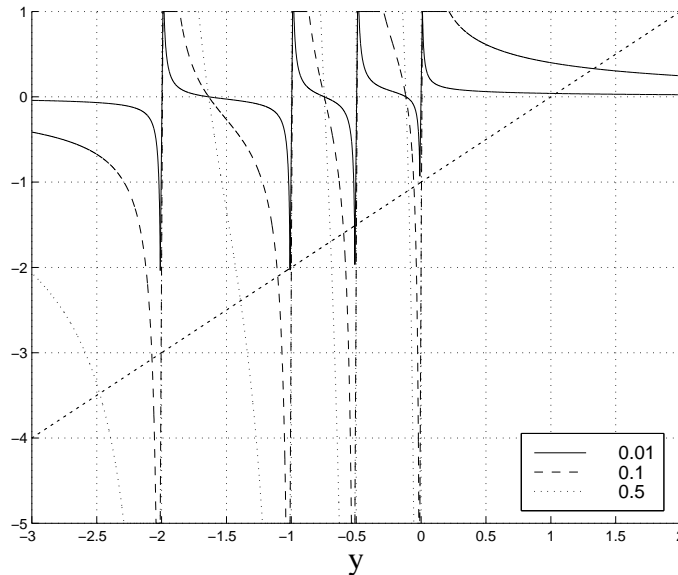


FIG. 1. Example of a graphic resolution of equation (71) where we have chosen  $\alpha^2 = (0.01, 0.1, 0.5)$  and  $\beta = -0.5$ .

## 2. Oscillation probability

Assuming that  $\alpha/|\beta| \ll 1$ , we can now expand (73) to get the following behaviours of the coefficients  $f_{x_i}^2$

$$f_{x_1}^2 \simeq 1 - \frac{\alpha^2}{\beta^2} \mathcal{I}(\beta^{-1}), \quad (85)$$

$$f_{x_{j>1}}^2 \simeq \frac{s_{j-1}}{(1 - \beta_{j-1})^2} \alpha^2. \quad (86)$$

$$(87)$$

One can easily check that, at this order,  $\sum f_{x_i}^2 = 1$ . Using the form (57) of the oscillation probability we deduce that

$$P(\gamma \rightarrow g) \simeq 4\alpha^2 \sum_{j \geq 1} \frac{s_j}{(1 - \beta_j)^2} \sin^2 \left( \frac{1 - \beta_j}{2} \Delta_\lambda u \right), \quad (88)$$

as announced in (63). It can be checked that the dominant contribution to the probability comes from the terms  $f_{x_1}^2 f_{x_j}^2$  in (57).

It is worth noting that the cut-off of the theory does not enter the result, due to the fact that in this special case all the sums are converging. Since  $\beta_j \leq 0$  and  $\alpha \ll 1$ , we are always in the weak mixing limit and the oscillation probability is well approximated by the sum of all the individual oscillation probabilities. The individual oscillation lengths are given by

$$\ell_{\text{osc}}^{(j)} = \frac{2\pi}{\Delta_\lambda} \frac{1}{1 - \beta_j} < \frac{2\pi}{\Delta_\lambda} = \ell_{\text{osc}}^{(1)}.$$

## B. $\beta > 0$

### 1. Eigenvalues

Since  $\beta_j = (j - 1)^2 \beta$ , one can easily show that the roots of (71) are such that

$$y_1 < 0, \quad \beta_i < y_{j+1} < \beta_{j+1} \quad (89)$$

with

$$y_1 \quad \text{solution of} \quad 1 - y = \pi \frac{\alpha^2}{\beta} \sqrt{\frac{\beta}{-y}} \coth \pi \sqrt{\frac{-y}{\beta}}, \quad (90)$$

$$y_{j>1} \quad \text{solutions of} \quad y - 1 = \pi \frac{\alpha^2}{\beta} \sqrt{\frac{\beta}{y}} \cot \pi \sqrt{\frac{y}{\beta}}. \quad (91)$$

In figure 2 we depict the graphic resolution of this equation. We introduce  $K$  the index of the closest  $\beta_i$  to unity. Contrary to the previous case, the discussion has to be split in four steps:

1.  $y_1$ : When  $-y_1/\beta \ll 1$ , (90) implies that  $1 - y_1 \simeq -\alpha^2/y_1$  so that  $y_1 \simeq -\alpha^2$  and the initial condition reduces to  $\alpha^2/\beta \ll 1$ . Thus when  $\alpha^2/\beta \ll 1$ , whatever the magnitude of  $\beta$ ,

$$y_1 \simeq -\alpha^2. \quad (92)$$

2.  $y_j, 1 < j < K$ : We set  $\sqrt{y_j/\beta} = (j - 1) - \epsilon_j$  with  $1 > \epsilon_j > 0$ , so that (91) rewrites as

$$1 - y_j = \frac{\pi \alpha^2}{\beta} \frac{\cot \pi \epsilon_j}{(j - 1) - \epsilon_j}. \quad (93)$$

Since the l.h.s. of (93) is positive we deduce that  $\epsilon_j < 1/2$ . Now, taking into account the fact that  $1 \in [(\beta_K + \beta_{K-1})/2, (\beta_K + \beta_{K+1})/2]$ , we deduce that  $1 - y_j > 1 - y_{K-1} > (\beta_K - \beta_{K-1})/2 = \beta(K - 3/2)/2$  and

thus (since  $K \geq 2$ ) that  $1 - y_j \geq \beta/4$ . If  $\alpha^2/\beta^2 \ll 1$ , then (93) implies that  $\cot \pi \epsilon_j \simeq 1/\pi \epsilon_j \ll 1$  from which we deduce  $\epsilon_j$  and then

$$y_{1 < j < K} \simeq \beta(j-1)^2 - \frac{2\alpha^2}{1 - \beta(j-1)^2}. \quad (94)$$

3.  $y_j, j > K+1$ : The argument follows the same lines as the previous one but we now set  $\sqrt{y_j/\beta} = (j-2) + \epsilon_j$  with  $1 > \epsilon_j > 0$ . We can now deduce from  $y_j - 1 \geq y_{K+2} - 1$  that  $y_j - 1 \geq \beta/2$  and then that if  $\alpha^2/\beta^2 \ll 1$ ,

$$y_{j > K+1} \simeq \beta(j-2)^2 + \frac{2\alpha^2}{\beta(j-2)^2 - 1}. \quad (95)$$

4.  $y_K, y_{K+1}$ : If  $K > 1$ , we set  $y_K = \beta_K(1 + \epsilon)$  and  $y_{K+1} = \beta_K(1 + \epsilon')$ , we can use the property of equation (91) to deduce, as before, that

$$0 < \epsilon < \frac{K-3/4}{(K-1)^2} \quad \text{and} \quad 0 < -\epsilon' < \frac{K-5/4}{(K-1)^2} \quad (96)$$

and then if  $\beta$  is small, we can conclude that  $\epsilon$  and  $\epsilon'$  are small compared to unity. Setting  $\delta \equiv \beta_K - 1$ ,  $\epsilon$  and  $\epsilon'$  are solution of (91) which reduces to

$$\delta + \epsilon \simeq \frac{2\alpha^2}{\epsilon} \quad \implies \quad 2\epsilon \simeq \delta \pm \sqrt{\delta^2 + 8\alpha^2}. \quad (97)$$

Now, if  $\delta \gg 2\sqrt{2}\alpha$ , we deduce that

$$y_K, y_{K+1} \in \left\{ \beta_K + \frac{2\alpha^2}{\beta_K - 1}, 1 - \frac{2\alpha^2}{\beta_K - 1} \right\}, \quad (98)$$

depending on the sign of  $\delta$  and with the constraint  $y_K < y_{K+1}$ . On the other hand, if  $\delta \ll 2\sqrt{2}\alpha$ ,

$$y_K \simeq 1 + \alpha\sqrt{2}, \quad y_{K+1} \simeq 1 - \alpha\sqrt{2}. \quad (99)$$

Note that if  $K = 1$ , then  $\beta \geq 2$  and the above discussion is still valid, but we just have the two classes of solutions  $y_1$  and  $y_{j>1}$ .

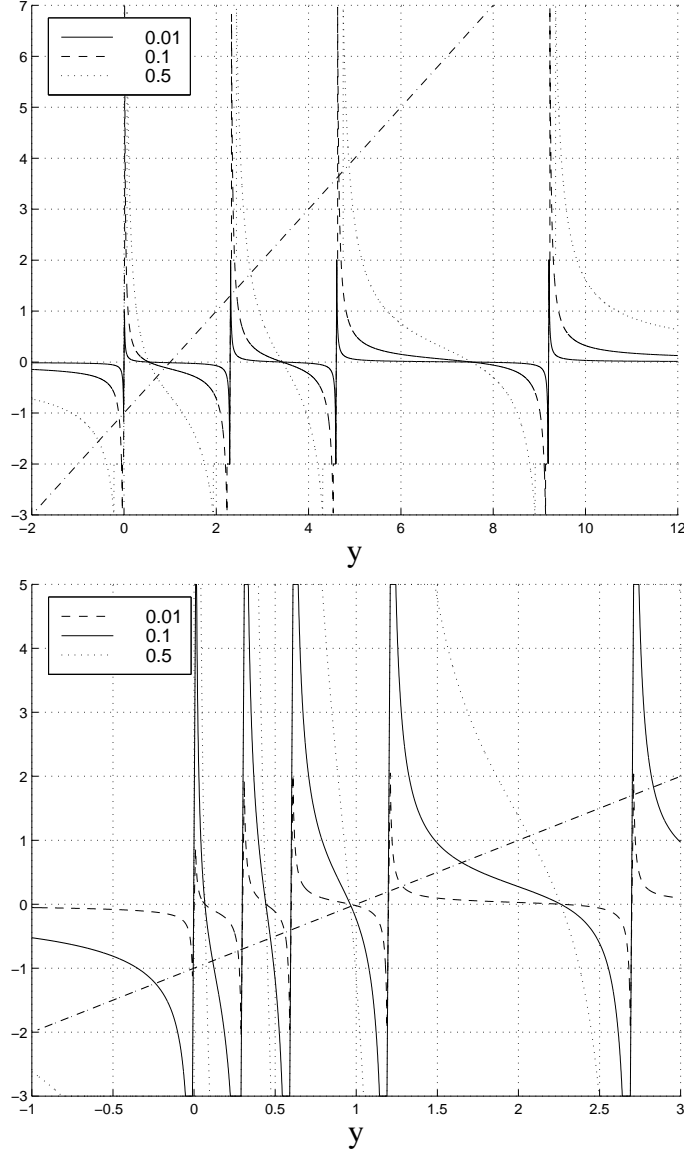


FIG. 2. Examples of a graphic resolution of equation (59) in the case  $\beta > 0$ . We have chosen  $\alpha^2 = (0.01, 0.1, 0.5)$ . The left plot describes the situation  $\beta = 2.3$  and the right plot the case where  $\beta = 0.3$ . In the latter case  $K = 4$ .

## 2. Oscillation probability

Assuming that  $\alpha/\beta \ll 1$ , we can expand (73) to get the following forms for the coefficients  $f_{x_i}^2$ . Assuming that  $\beta_K - 1 < 0$ ,

$$f_{x_1 \leq j < K}^2 \simeq \frac{s_j}{(1 - \beta_j)^2} \alpha^2 \quad (100)$$

$$f_{x_K}^2 \simeq \begin{cases} \frac{1}{2} & (|\beta_K - 1| \ll 2\sqrt{2}\alpha) \\ \frac{s_K}{(1 - \beta_K)^2} \alpha^2 & (|\beta_K - 1| \gg 2\sqrt{2}\alpha) \end{cases} \quad (101)$$

$$f_{x_{K+1}}^2 \simeq \begin{cases} \frac{1}{2} & (|\beta_K - 1| \ll 2\sqrt{2}\alpha) \\ 1 - \frac{\alpha^2}{\beta^2} \mathcal{I}(\beta^{-1}) & (|\beta_K - 1| \gg 2\sqrt{2}\alpha) \end{cases} \quad (102)$$

$$f_{x_j > K+1}^2 \simeq \frac{s_{j-1}}{(1 - \beta_{j-1})^2} \alpha^2. \quad (103)$$

When  $\beta_K - 1 > 0$ ,  $f_{x_{K-1}}^2$  is then given by (102) and  $f_{x_{K+1}}^2$  is of the form (103).



It can be checked that, at lowest order  $\sum f_{x_i}^2 = 1$ . One can further check that when  $|\beta_K - 1| \ll 2\sqrt{2}\alpha$  the first correction to (101–102) is  $-\alpha^2 \mathcal{I}(\beta^{-1})/2\beta^2$  and that  $\sum f_{x_i}^2 = 1$  is also satisfied (which indeed has to be order by order). Now, the oscillation probability (57) reduces to

$$P(\gamma \rightarrow g) \simeq \begin{cases} (1 - \epsilon) \sin^2 \left[ \Delta_M s_K^{1/2} u \right] + 4\alpha^2 \sum_{i \neq K} \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right] & (|\beta_K - 1| \ll 2\sqrt{2}\alpha) \\ 4\alpha^2 \sum_{i \geq 1} \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right] & (|\beta_K - 1| \gg 2\sqrt{2}\alpha), \end{cases} \quad (104)$$

with  $\eta \equiv \alpha^2 \mathcal{I}(\beta^{-1})/\beta^2$  as announced in (63). Thus, when  $|\beta_K - 1| \gg \alpha$ , we are in a weak mixing regime and the probability is obtained by summing over all the individual probabilities. Otherwise, we are in a regime of strong mixing with the state  $K$ , and the oscillation length with this state is now given by

$$\ell_{\text{osc}} = \frac{\pi}{\Delta_M s_K^{1/2}}.$$

### C. Summary and discussion

In the limit where  $\alpha^2 \ll 1$ , we have estimated the oscillation probability (57) to be

- $\beta < 0$ :

$$P(\gamma \rightarrow g) \simeq 4\alpha^2 \sum_{i \geq 1} \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right],$$

- $\beta > 0$ :

$$P(\gamma \rightarrow g) \simeq \begin{cases} I \sin^2 \left[ \Delta_M s_K^{1/2} u \right] + 4\alpha^2 \sum_{i \neq K} \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right] & (|\beta_K - 1| \ll 2\sqrt{2}\alpha) \\ 4\alpha^2 \sum_{i \geq 1} \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right] & (|\beta_K - 1| \gg 2\sqrt{2}\alpha^2), \end{cases}$$

with  $I \equiv 1 - \alpha^2 \mathcal{I}(\beta^{-1})/\beta^2$ , as announced in (61) and (63). When  $|\beta_j - 1| \gg 2\sqrt{2}\alpha^2$  for all  $j$ , it is interesting to study the limit where  $u \gg \ell_{\text{osc}}$  in which case we can assume that the sines can be replaced by their average value to get the two limiting behaviours:

$$P(\gamma \rightarrow g) \simeq 2\alpha^2 \left( 1 + \frac{\pi^4}{45\beta^2} \right) \quad (|\beta| \gg 1), \quad (105)$$

$$P(\gamma \rightarrow g) \simeq \pi \frac{\alpha^2}{\sqrt{|\beta|}} \quad (|\beta| \ll 1). \quad (106)$$

As explained in § IV, we respectively see on (105–106) the *small* and *large* radius regimes where the extra–dimensions either have no effect (105) or enhance (106) the probability. We also find the regime of strong mixing with the  $K^{\text{th}}$  KK graviton and the effect on the oscillation length as discussed below equation (62).

## VI. ESTIMATION OF THE PROBABILITY IN A SIX DIMENSIONAL SPACETIME

Let us now turn to the physically more interesting case of  $n = 2$  extra–dimensions. Now, equation (59) cannot be solved exactly in general, but its solutions can be well approximated when the coupling between the photon and the graviton (this coupling is measured by  $\Delta_M$  in the matrix (26)) is small enough compared to the typical mass parameters of the mixing particles [i.e. the diagonal terms in (26)]. We follow the same lines as in the previous section.

$$\mathbf{A}. 1 \geq \beta > 0$$

### 1. Eigenvalues

In that case, the plasma effects dominate over the vacuum polarisation in  $\Delta_\lambda$ ; all the  $\beta_i$  are positive and one can easily show that the roots of (59) are such that

$$y_1 < 0, \quad y_i \in ]\beta_{i-1}, \beta_i[, \quad y_{N_D+1} > \beta_{N_D}. \quad (107)$$

We introduce  $k(i)$  the index such that  $\beta_{k(i)}$  is the closest  $\beta_i$  to  $y_i$ . From (107) one has that  $k(i) \in \{i-1, i\}$ . The eigenvalue equation (59) for the root  $y_i$  can be rewritten as

$$y_i - 1 = \alpha^2 \frac{s_k}{y_i - \beta_k} + \mathcal{F}_k(y_i), \quad (108)$$

where  $\mathcal{F}_k$  is defined as

$$\mathcal{F}_k(y) \equiv \alpha^2 \sum_{j \neq k} \frac{s_j}{y - \beta_j}. \quad (109)$$

$\mathcal{F}$  with no subscript denotes the function defined by the sum (109) taken over all indices  $j$  from  $j = 1$  to  $j = N_D$ . To finish, we introduce the index  $K$  such that  $\beta_K$  is the closest  $\beta_i$  to unity and then,

$$\forall i \neq K \quad |\beta_i - 1| \geq \frac{\beta}{2}. \quad (110)$$

As in the five dimensional case, the determination of the roots  $y_i$  has to be split in the three following cases:

- $i \leq K - 1$ : We first show that  $k(i) = i$ . For that purpose, we consider the function  $\mathcal{H}(y)$  defined by

$$\mathcal{H}(y) \equiv 1 - y + \mathcal{F}(y). \quad (111)$$

This function is strictly decreasing on  $]\beta_{i-1}, \beta_i[$  ( $\mathcal{H}$  vanishes only once in this interval in  $y = y_i$ ). Showing that  $\mathcal{H}\left(\frac{\beta_i + \beta_{i-1}}{2}\right) \geq 0$ , is then enough to prove that  $k(i) = i$ . Since one has

$$1 - \frac{\beta_i + \beta_{i-1}}{2} \geq \frac{\beta}{2} \quad \text{and} \quad \forall k \quad \left| \frac{\beta_i + \beta_{i-1}}{2} - \beta_k \right| \geq \frac{\beta}{2}, \quad (112)$$

using (C19), one obtains

$$\left| \mathcal{F}\left(\frac{\beta_i + \beta_{i-1}}{2}\right) \right| \leq \mathcal{Q} \frac{\alpha^2}{\beta^2} \sup\left(\mathcal{Q}', \sqrt{\beta y}\right) \leq \mathcal{Q} \mathcal{Q}' \frac{\alpha^2}{\beta^2} \quad \text{for } y \leq 1 \text{ and } \beta \leq 1. \quad (113)$$

(the constants  $\mathcal{Q}$  and  $\mathcal{Q}'$  are defined in equation (C19) of appendix C). Comparing (112) and (113) one sees that for  $\alpha$  smaller enough than  $\beta$  (namely  $2\mathcal{Q}\mathcal{Q}'\frac{\alpha^2}{\beta^3} < 1$ ), one has  $\mathcal{H}\left(\frac{\beta_i + \beta_{i-1}}{2}\right) > 0$  and then that  $k(i) = i$ .

We will assume in the following the slightly stronger constraint

$$10\mathcal{Q}\mathcal{Q}'\frac{\alpha^2}{\beta^3} < 1. \quad (114)$$

Now, we set  $y_i = \beta_i - \epsilon_i$ , with  $\epsilon_i > 0$ . Equation (108) can be rewritten as an equation for  $\epsilon_i$ , with  $\mathcal{F}_i \equiv \mathcal{F}_i(y_i)$ ,

$$\frac{\epsilon_i^2}{\beta_i - 1 - \mathcal{F}_i} - \epsilon_i - \frac{\alpha^2 s_i}{\beta_i - 1 - \mathcal{F}_i} = 0, \quad (115)$$

the positive solution of which is given to leading order by

$$\epsilon_i \simeq \frac{\alpha^2 s_i}{1 - \beta_i}, \quad (116)$$

when  $\alpha$  and  $\beta$  verify the constraint (114)<sup>2</sup>. The eigenvalues are then given at leading order by

$$y_i \simeq \beta_i - \alpha^2 \frac{s_i}{1 - \beta_i}. \quad (117)$$

- $i \geq K+2$ : Using a similar line of reasoning as in the previous case, one can show that  $k(i) = i-1$  and then that  $y_i$  is given at dominant order by (for  $\alpha$  and  $\beta$  verifying (114))

$$y_i \simeq \beta_{i-1} + \alpha^2 \frac{s_{i-1}}{\beta_{i-1} - 1}. \quad (118)$$

- $i = K, K+1$ : We first estimate the root  $y_K$ . We assume that  $1 \in [\beta_K, \beta_{K+1}[$  (similar conclusions can be obtained when  $1 \in [\beta_{K-1}, \beta_K[$ ), we have

$$1 - \frac{\beta_K + \beta_{K-1}}{2} > \frac{\beta}{2}. \quad (119)$$

As in the previous case, this is enough to show that  $y_K \in [\beta_{K-1} + \frac{\beta_K - \beta_{K-1}}{2}, \beta_K[$ <sup>3</sup>. We set  $y_K = \beta_K - \epsilon_K$  and  $\mathcal{F}_K \equiv \mathcal{F}_K(y_K)$  with  $\epsilon_K > 0$  solution of

$$\frac{\epsilon_K^2}{\beta_K - 1 - \mathcal{F}_K} - \epsilon_K - \frac{\alpha^2 s_K}{\beta_K - 1 - \mathcal{F}_K} = 0, \quad (120)$$

the positive root of which is

$$\epsilon_K = \frac{1 - \beta_K + \mathcal{F}_K}{2} \left( -1 + \sqrt{1 + \frac{4\alpha^2 s_K}{(\beta_K - 1 - \mathcal{F}_K)^2}} \right). \quad (121)$$

For  $\alpha$  smaller enough than  $\beta$ <sup>4</sup>, one can consider the two limiting regimes (we will not consider here the intermediate case, in order to simplify the discussion)

$$\epsilon_K \simeq \begin{cases} \alpha\sqrt{s_K} & \text{if } |\beta_K - 1| \ll 2\alpha\sqrt{s_K} \\ \frac{\alpha^2 s_K}{1 - \beta_K} & \text{if } |\beta_K - 1| \gg 2\alpha\sqrt{s_K}. \end{cases} \quad (122)$$

Let us now turn to the evaluation of the root  $y_{K+1}$ . The discussion mimics the previous one. Assuming that 1 is not too close to  $\frac{\beta_K + \beta_{K+1}}{2}$ <sup>5</sup>, one can show that  $y_{K+1} \in ]\beta_K, \beta_K + \frac{\beta_{K+1} - \beta_K}{2}[$  under the condition (114). Then we write  $y_{K+1} = \beta_K - \epsilon_{K+1}$ , with  $\epsilon_{K+1} < 0$ .  $\epsilon_{K+1}$  is solution of equation (120) with  $\mathcal{F}_K \equiv \mathcal{F}_K(y_{K+1})$ . One considers (under the condition on  $\alpha$  and  $\beta$  of footnote 4) the two limiting regimes

$$y_{K+1} \simeq \begin{cases} \beta_K + \alpha\sqrt{s_K} & \text{for } |\beta_K - 1| \ll 2\alpha\sqrt{s_K}, \\ 1 + \mathcal{F}(1) & \text{for } |\beta_K - 1| \gg 2\alpha\sqrt{s_K}. \end{cases} \quad (123)$$

<sup>2</sup>to establish this we have used (C19) and (C21) and showed that (114) leads to  $|\beta_i - 1| \gg \mathcal{F}_i$  and  $|\beta_i - 1| \gg 2\alpha\sqrt{s_i}$  which in turn leads to the expression (116). In the rest of this section  $a \gg b$  means that  $a > 10b$  which we assumed to be enough to neglect  $b$  with respect to  $a$ .

<sup>3</sup>here we use (114) again.

<sup>4</sup>One has here to impose a slightly stronger condition than the previous one: namely  $4\mathcal{Q}\mathcal{Q}' \frac{\alpha}{\beta^2} < 1$  in order to be able to neglect  $\mathcal{F}_K$  with respect to  $2\alpha\sqrt{s_K}$ . This also insures that the expressions (122) are valid.

<sup>5</sup>namely  $\left| 1 - \frac{\beta_K + \beta_{K+1}}{2} \right| > \frac{\beta}{10}$ . When this is not the case, one obtains the same results as in (123) when  $|\beta_K - 1| \gg 2\alpha\sqrt{s_K}$ .

## 2. Oscillation probability

We now need to expand the coefficients (58) in order to estimate the probability (57).

- For  $i \leq K-1$ , we have from equation (58) and (117)

$$f_{x_i}^2 = \left[ 1 + \frac{(1 - \beta_i)^2}{\alpha^2 s_i} + \mathcal{G}_i \right]^{-1} \quad (124)$$

with

$$\mathcal{G}_i \equiv \mathcal{G}_i(y_i) \equiv \alpha^2 \sum_{k=1, k \neq i}^{N_D} \frac{s_k}{(y_i - \beta_k)^2}. \quad (125)$$

Using (C20), one can then show that under the condition (114)

$$\frac{(1 - \beta_i)^2}{\alpha^2 s_i} \gg \max(1, \mathcal{G}_i) \quad (126)$$

so that, at dominant order,

$$f_{x_i}^2 \simeq \alpha^2 \frac{s_i}{(1 - \beta_i)^2}. \quad (127)$$

- For  $i \geq K+2$ , we find in a similar way (and under the same condition)

$$f_{x_i}^2 \simeq \alpha^2 \frac{s_{i-1}}{(1 - \beta_{i-1})^2}. \quad (128)$$

- For  $i = K, K+1$ , we distinguish the two regimes  $|\beta_K - 1| \ll 2\alpha\sqrt{s_K}$  and  $|\beta_K - 1| \gg 2\alpha\sqrt{s_K}$ .

Assuming that  $|\beta_K - 1| \gg 2\alpha\sqrt{s_K}$ , we find <sup>6</sup> the dominant contribution to  $f_{x_K}$  and  $f_{x_{K+1}}$  to be

$$f_{x_K}^2 \simeq \alpha^2 \frac{s_K}{(1 - \beta_K)^2} \quad \text{and} \quad f_{x_{K+1}}^2 \simeq 1. \quad (129)$$

When  $|\beta_K - 1| \ll 2\alpha\sqrt{s_K}$ , the dominant contribution to  $f_{x_K}^2$  and  $f_{x_{K+1}}^2$  are

$$f_{x_K}^2 \simeq \frac{1}{2} \quad \text{and} \quad f_{x_{K+1}}^2 \simeq \frac{1}{2}. \quad (130)$$

Now, inserting these results in (57), we find that the oscillation probability is given to dominant order, for  $|\beta_K - 1| \gg 2\alpha\sqrt{s_K}$ , by

$$P(\gamma \rightarrow g) \simeq 4 \sum_{i \neq K+1} f_{x_i}^2 \sin^2 \left[ \frac{x_i - x_{K+1}}{2} u \right] \simeq 4 \sum_{i=1}^{i=N_D} \alpha^2 \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right] \quad (131)$$

This expression, as announced, is analogous to (63) and corresponds to the case when no KK state mixes strongly with the photon.

Now, when  $|\beta_K - 1| \ll 2\alpha\sqrt{s_K}$ , one has

---

<sup>6</sup>Using (C20), we show that this expansion is valid under the condition of the footnote (4).

$$P(\gamma \rightarrow g) \simeq (1 - \eta) \sin^2 \left[ \frac{x_K - x_{K+1}}{2} u \right] + 2 \sum_{P=K, K+1} \sum_{i \neq P} f_{x_i}^2 \sin^2 \left[ \frac{x_i - x_P}{2} u \right] \quad (132)$$

which can be rewritten as

$$P(\gamma \rightarrow g) \simeq (1 - \eta) \sin^2 [\Delta_M \sqrt{s_K} u] + 4 \sum_{i \neq K} \alpha^2 \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{(1 - \beta_i)}{2} \Delta_\lambda u \right] \quad (133)$$

with  $\eta \equiv \sum_{i \neq K} \alpha^2 s_i / (1 - \beta_i)^2 \ll 1$ . As in the five dimensional case it was obtained by imposing that  $\sum f_{x_i}^2 = 1$  order by order. This corresponds to the case where one KK state mixes strongly with the photon and again the oscillation probability is found to be equivalent to (61).

## B. $-1 \leq \beta < 0$

We now consider the case where  $\beta < 0$  (i.e. when the vacuum contribution dominates over the plasma effects in  $\Delta_\lambda$ ). It is easy to see graphically (see figure 1) that the  $N_D + 1$  solutions of the eigenvalue equation (59) are such that

$$y_1 > 1, \quad y_{i+1} \in ]\beta_{i+1}, \beta_i[, \quad y_{N_D+1} < \beta_{N_D}. \quad (134)$$

### 1. Eigenvalues

We do not detail the computation of the eigenvalues since it is similar to the former case. It is even simpler since now we do not have to single out the mode  $K$  (look for instance to the five dimensional case § V A).

We have only to assume the less stringent constraint than (114),  $10 \frac{\alpha^2}{\beta^2} \mathcal{Q} \mathcal{Q}' < 1$ , in order for the following expansions to be valid. Under this condition one can show that

$$\begin{aligned} y_1 &\simeq 1 + \alpha^2 \sum_i \frac{s_i}{1 - \beta_i}, \\ y_{j>1} &\simeq \beta_{j-1} - \frac{s_{j-1}}{1 - \beta_{j-1}} \alpha^2. \end{aligned} \quad (135)$$

### 2. Oscillation Probability

The coefficients (58) are then given at leading order by

$$\begin{aligned} f_{x_1}^2 &\simeq 1, \\ f_{x_{j>1}}^2 &\simeq \alpha^2 \frac{s_{j-1}}{(1 - \beta_{j-1})^2}. \end{aligned} \quad (136)$$

The oscillation probability (57) can be expanded as

$$P(\gamma \rightarrow g) \simeq 4 f_{x_1}^2 \sum_{j>1} f_{x_j}^2 \sin^2 \left[ \frac{x_1 - x_j}{2} u \right] \quad (137)$$

where we have neglected higher order terms. This can be rewritten as

$$P(\gamma \rightarrow g) \simeq 4 \alpha^2 \sum_j \frac{s_j}{(1 - \beta_j)^2} \sin^2 \left[ \frac{1 - \beta_j}{2} \Delta_\lambda u \right] \quad (138)$$

which is again analogous to (63).

### C. $|\beta| > 1$

We introduce the new parameters  $\bar{\alpha} \equiv \Delta_M/\Delta_m < 0$  and  $\bar{\beta}_i \equiv \Delta_m^{(r_i)}/\Delta_m = \bar{p}_{(r_i)}^2$ ,  $\bar{\gamma} \equiv \Delta_\lambda/\Delta_m$  and  $z \equiv x/\Delta_m$ . The eigenvalue equation (A9) can be rewritten as

$$z - \bar{\gamma} = \bar{\alpha}^2 \sum_{i=1}^{N_D} \frac{s_i}{(z - \bar{\beta}_i)} = \frac{\bar{\alpha}^2}{z} + \bar{\alpha}^2 \sum_{i=2}^{N_D} \frac{s_i}{(z - \bar{\beta}_i)}. \quad (139)$$

One sees easily graphically that this equation admits one negative root  $z_1$  and that the other  $z_i$  ( $i \geq 3$ ) verify  $z_i \in ]\bar{\beta}_{i-1}, \bar{\beta}_i[$ . We discuss here only the case where  $\bar{\gamma}$  is closer to 0 than to 1 (we assume that  $\bar{\gamma} < 1/4$ ). The discussion is very similar to the previous cases. Under the condition  $10\bar{\alpha}^2 Q Q' < 1$ , one finds that the roots  $z_i$  with  $i \geq 2$  are given by

$$z_i \simeq \bar{\beta}_{i-1} + \frac{\bar{\alpha}^2 s_{i-1}}{\bar{\beta}_{i-1} - \bar{\gamma}} \quad \text{so that} \quad f_{x_i}^2 \simeq \frac{\bar{\alpha}^2 s_{i-1}}{(\bar{\beta}_{i-1} - \bar{\gamma})^2}. \quad (140)$$

Under the condition  $4\bar{\alpha} Q Q' < 1$ , the two roots  $z_1$  and  $z_2$  are given, for  $\bar{\gamma} > 0$  by<sup>7</sup> by

$$\begin{cases} z_1 \simeq \bar{\alpha} \\ z_2 \simeq -\bar{\alpha} \end{cases} \quad \text{if } \bar{\gamma} \ll 2|\bar{\alpha}| \quad \text{and by} \quad \begin{cases} z_1 \simeq -\frac{\bar{\alpha}^2}{\bar{\gamma}} \\ z_2 \simeq \bar{\gamma} + \mathcal{F}(\bar{\gamma}) \end{cases} \quad \text{if } \bar{\gamma} \gg 2|\bar{\alpha}| \quad (141)$$

from which we deduce that the coefficients are given either by

$$\begin{cases} f_{x_1}^2 \simeq \frac{1}{2} \\ f_{x_2}^2 \simeq \frac{1}{2} \end{cases} \quad \text{if } \bar{\gamma} \ll 2|\bar{\alpha}| \quad \text{or by} \quad \begin{cases} f_{x_1}^2 \simeq 1 \\ f_{x_2}^2 \simeq \frac{\bar{\alpha}^2}{\bar{\gamma}^2} \end{cases} \quad \text{if } \bar{\gamma} \gg 2|\bar{\alpha}|. \quad (142)$$

When  $\bar{\gamma} \ll 2|\bar{\alpha}|$ , the oscillation probability is given by

$$P(\gamma \rightarrow g) \simeq (1 - \bar{\eta}) \sin^2(\Delta_M u) + 4 \sum_{i \geq 2} \frac{\alpha^2 s_i}{\beta_i^2} \sin^2 \left[ \frac{\beta_i}{2} \Delta_\lambda u \right] \quad (143)$$

and when  $\bar{\gamma} \gg 2|\bar{\alpha}|$  by

$$P(\gamma \rightarrow g) \simeq 4 \sum_{i \geq 1} \frac{\alpha^2 s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right]. \quad (144)$$

The small coefficient  $\bar{\eta} \equiv \sum_{i \geq 2} s_i \alpha^2 / \beta_i^2$  is obtained as in (132).

### D. Summary

In the limit where  $\alpha^2 < 1$  we have estimated the oscillation probability (57) for a six dimensional spacetime to be

- $1 \geq \beta > 0$ : For  $|\beta_K - 1| \gg 2\alpha\sqrt{s_K}$ ,

$$P(\gamma \rightarrow g) \simeq 4 \sum_{i=1}^{i=N_D} \alpha^2 \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right] \quad (145)$$

and for  $|\beta_K - 1| \ll 2\alpha\sqrt{s_K}$

$$P(\gamma \rightarrow g) \simeq (1 - \eta) \sin^2[\Delta_M \sqrt{s_K} u] + 4 \sum_{i \neq K} \alpha^2 \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{(1 - \beta_i)}{2} \Delta_\lambda u \right], \quad (146)$$

these results being valid as long as  $4Q Q' \frac{\alpha}{\beta^2} < 1$  and the coefficient  $\eta$  being defined in equation (133).

---

<sup>7</sup>For  $\bar{\gamma} < 0$  the results are similar; one has only to exchange the expressions of  $z_1$  and  $z_2$  in the case  $|\bar{\gamma}| \gg 2|\bar{\alpha}|$ .

- $-1 \leq \beta < 0$ :

$$P(\gamma \rightarrow g) \simeq 4\alpha^2 \sum_j \frac{s_j}{(1 - \beta_j)^2} \sin^2 \left[ \frac{1 - \beta_j}{2} \Delta_\lambda u \right], \quad (147)$$

valid if  $10 \frac{\alpha^2}{\beta^2} \mathcal{Q}\mathcal{Q}' < 1$ .

- $|\beta| > 1$ :

$$\begin{aligned} P(\gamma \rightarrow g) &\simeq (1 - \tilde{\eta}) \sin^2(\Delta_M u) + 4 \sum_{i \geq 2} \frac{\alpha^2 s_i}{\beta_i^2} \sin^2 \left[ \frac{\beta_i}{2} \Delta_\lambda u \right] \\ &\simeq 4 \sum_{i \geq 1} \frac{\alpha^2 s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{1 - \beta_i}{2} \Delta_\lambda u \right] \end{aligned} \quad (148)$$

respectively for  $\bar{\gamma} \ll 2|\bar{\alpha}|$  and for  $\bar{\gamma} \gg 2|\bar{\alpha}|$ , the result being valid if  $10\bar{\alpha}^2 \mathcal{Q}\mathcal{Q}' < 1$  and the small coefficient  $\tilde{\eta}$  is defined in (143).

## VII. MIXING IN AN INHOMOGENEOUS FIELD

In all the previous sections, we have assumed that the magnetic field was homogeneous. This is however a very crude approximation for most of the realistic physical systems. In this section we first extend our analysis to inhomogeneous magnetic fields and give some implications of the inhomogeneity of the external field.

### A. Computation of the oscillation probability

Following [4], we rewrite the equation of evolution (25) as a Schrödinger equation

$$i\partial_u \vec{\mathcal{V}} = (\mathcal{H}_0 + \mathcal{H}_1) \vec{\mathcal{V}}, \quad (149)$$

where we set  $\vec{\mathcal{V}} \equiv (A, G^{(0)}, \dots, G^{(N)})$ . The two matrices  $\mathcal{H}_0$  and  $\mathcal{H}_1$  are respectively defined by

$$\mathcal{H}_0(u) \equiv \omega + \begin{pmatrix} \Delta_\lambda & & & \\ & 0 & & \\ & & \Delta_m^{(1)} & \\ & & & \ddots \\ & & & & \Delta_m^{(N)} \end{pmatrix} \quad (150)$$

and

$$\mathcal{H}_1(u) \equiv \begin{pmatrix} 0 & \Delta_M & \cdots & \Delta_M \\ \Delta_M & & & \\ \vdots & & & \\ \Delta_M & & & \end{pmatrix}. \quad (151)$$

We assume that  $\mathcal{H}_1$  is a perturbation compared to  $\mathcal{H}_0$ . This approximation is equivalent to saying that  $\Delta_M/\Delta_\lambda$  and  $\Delta_M/\Delta_m$  are small compared to unity, i.e. that  $\alpha \ll 1$  and  $\alpha/\beta \ll 1$ . When  $H_0$  is inhomogeneous only  $\Delta_M$  and  $\Delta_\lambda$  depend on  $u$  while all the  $\Delta_m^{(q)}$  are constant.

We first solve (149) at zeroth order, i.e. by neglecting  $\mathcal{H}_1$  with respect to  $\mathcal{H}_0$ , as

$$\vec{\mathcal{V}}^{(0)}(u) = U(u) \vec{\mathcal{V}}^{(0)}(0), \quad (152)$$

where the evolution operator  $U$  is defined by

$$U(u) \equiv \exp -i \int_0^u \mathcal{H}_0(u') du'. \quad (153)$$

Note that at this order there is no mixing effect since  $\mathcal{H}_0$  is diagonal.

The general solution of (149) is obtained by shifting to the “interaction representation” where  $\vec{\mathcal{V}}_{\text{int}} \equiv U^\dagger \vec{\mathcal{V}}$  so that (149) can be rewritten as

$$i\partial_u \vec{\mathcal{V}}_{\text{int}} = \mathcal{H}_{\text{int}} \vec{\mathcal{V}}_{\text{int}} \quad (154)$$

with  $\mathcal{H}_{\text{int}} \equiv U^\dagger \mathcal{H}_1 U$ . This equation can be solved iteratively by setting  $\vec{\mathcal{V}}_{\text{int}} = \sum \vec{\mathcal{V}}_{\text{int}}^{(k)}$  with

$$\vec{\mathcal{V}}_{\text{int}}^{(n+1)} = -i \int_0^u du' \mathcal{H}_{\text{int}}(u') \vec{\mathcal{V}}_{\text{int}}^{(n)}(u'), \quad (155)$$

with the initial condition  $\vec{\mathcal{V}}_{\text{int}}^{(0)} \equiv \vec{\mathcal{V}}(0)$ .

Introducing the basis  $\{\vec{\mathcal{A}}, \vec{\mathcal{G}}_q\}$  with  $\vec{\mathcal{A}} \equiv (1, 0, \dots, 0)$  and  $\vec{\mathcal{G}}_q$  being the state of the  $q^{\text{th}}$  graviton,  $\mathcal{G}_q^i = \delta_q^i$  for  $i \in \{1, \dots, N+2\}$  and starting with an initial state describing a pure photon, i.e.  $\vec{\mathcal{V}}(0) = A(0)\vec{\mathcal{A}}$ , we obtain

$$\vec{\mathcal{V}}_{\text{int}}^{(1)}(u) = -i \int_0^u dz \Delta_M(z) \sum_q e^{i \int_0^z (\Delta_m^{(q)} - \Delta_\lambda(y)) dy} A(0) \vec{\mathcal{G}}_q, \quad (156)$$

where we have used that  $\mathcal{H}_0 \vec{\mathcal{A}} = \Delta_\lambda \vec{\mathcal{A}}$ ,  $\mathcal{H}_1 \vec{\mathcal{A}} = \Delta_M \sum_q \vec{\mathcal{G}}_q$  and  $\mathcal{H}_1 \vec{\mathcal{G}}_q = \Delta_m^{(q)} \vec{\mathcal{G}}_q$ . If we restrict to the first iteration, the oscillation probability is then given by

$$P(\gamma \rightarrow g) = \sum_q \left| \langle \vec{\mathcal{G}}_q | \vec{\mathcal{V}}^{(0)} + \vec{\mathcal{V}}^{(1)} \rangle \right|^2 = \sum_q \left| \langle \vec{\mathcal{G}}_q | \vec{\mathcal{V}}_{\text{int}}^{(1)} \rangle \right|^2, \quad (157)$$

that is, using (156),

$$P(\gamma \rightarrow g) = \sum_q \left| \int_0^u \Delta_M(u') e^{i \Delta_m^{(q)} u' - i \int_0^{u'} \Delta_\lambda(u'') du''} du' \right|^2. \quad (158)$$

We can check that in a homogeneous field we recover (63), i.e. the oscillation probability in the weak mixing case. Note that in the particular case of the weak mixing this method of computing the oscillation probability is shorter than the one used in the two former sections since it does not involve the determination of the eigenvalues of  $\mathcal{M}$ . But, one has to assume that the probability is small compared to unity [4], which is not necessarily the case for instance when we are in the strong mixing regime.

## B. Example of applications

As an example, we consider the mixing in a periodic magnetic field of the form  $H_0 \cos \Delta_0 u$  with  $\Delta_0 > 0$  for which the oscillation probability, in the weak mixing regime, is given by (158)

$$P(\gamma \rightarrow g) = \sum_{i \geq 1} s_i \left| \int_0^u dz \Delta_M \cos(\Delta_0 z) e^{i \Delta_m^{(r_i)} z} e^{-i \int_0^z \Delta_\lambda(v) dv} \right|^2. \quad (159)$$

Assume that  $\Delta_{\text{plasma}}$  dominates so that we can neglect the variation of  $\Delta_\lambda$  with  $z$  then, the probability becomes

$$P(\gamma \rightarrow g) \simeq \Delta_M^2 \sum_i s_i \left( \frac{1}{(\Delta_m^{(r_i)} - \Delta_\lambda^{(-)})^2} \sin^2 \left[ \frac{\Delta_m^{(r_i)} - \Delta_\lambda^{(-)}}{2} z \right] + \frac{1}{(\Delta_m^{(r_i)} - \Delta_\lambda^{(+)})^2} \sin^2 \left[ \frac{\Delta_m^{(r_i)} - \Delta_\lambda^{(+)}}{2} z \right] \right) \quad (160)$$

where we have kept only the resonant term, which depends on the sign of  $\Delta_m^{(r_i)} - \Delta_\lambda$ , and where we have defined

$$\Delta_\lambda^{(\pm)} \equiv \Delta_\lambda \pm \Delta_0. \quad (161)$$



Now if  $|\Delta_m^{(r_i)} - \Delta_\lambda^{(+)}| \ll |\Delta_M|$  or  $|\Delta_m^{(r_i)} - \Delta_\lambda^{(-)}| \ll |\Delta_M|$  we find a strong mixing regime, meaning that because of the resonance there will exist a mode for which the probability is enhanced.

In conclusion, the important scale that fixes the photon effective mass is now  $\Delta_\lambda^{(\pm)} \sim \Delta_0$  if we are in a regime where  $|\Delta_\lambda| \ll \Delta_0$ . We can then have the same discussion as in the previous sections but with  $\beta$  defined as

$$\beta = \frac{\Delta_m}{\Delta_\lambda^{(\pm)}}, \quad (162)$$

according to the sign of  $\Delta_\lambda$ . The length scale  $\lambda_\gamma$  is now given by  $(\omega\Delta_0)^{-1/2}$  and it follows that we expect the two following effects:

1. by increasing  $\Delta_0$  we can hope to make  $\beta$  as small as wanted and thus to get a large enhancement of the oscillation probability. What happens is that the scale  $\Delta_\lambda$  is replaced by  $\Delta_0$  and thus that a departure from the four dimensional case will be observed if  $(\omega\Delta_0)^{-1/2} < R$ . When the field is homogeneous, the scale  $\Delta_\lambda^{-1}$  is usually very large compared to  $R$  (see § VIII and § IX), which implies that there is little hope to see any effect of the extra-dimensions. By using an inhomogeneous field, we change the scale associated with the photon effective mass which is now governed by  $\Delta_0^{-1}$  that can be tried to be lowered to a scale close to  $R$ .
2. whatever the sign of  $\Delta_\lambda$ , we expect to have strong mixing occurring for all values of  $\Delta_0$  such that

$$|\Delta_m^{(r_i)} - (\Delta_\lambda \pm \Delta_0)| \ll \Delta_M.$$

By varying slowly  $\Delta_0$  or  $\omega$ , we expect to see a series of strong and weak mixing regimes.

The amplitude of these two effects will be discussed in the last section of this article.

## VIII. APPLICATION TO ASTROPHYSICS AND COSMOLOGY

Magnetic fields are observed in most astrophysical systems but the origin of galactic and cosmological magnetic fields is still unknown [44]. A possibility is that these fields have a primordial origin since such a magnetic field can be generated in a number of early universe mechanisms [45] such as in collisions of bubbles produced in a first order phase transition [46] or during an inflationary phase [47].

The efficiency of the photon-graviton and of the photon-axion mixing depends both on the value of the magnetic field and on the spatial extension of this field,  $\Lambda_c$  say. We study the order of magnitude of these mixings on the cosmic microwave background, on pulsars and magnetars.

The required quantities for our discussion are  $\Delta_M$ ,  $\Delta_m$ ,  $\Delta_{\text{plasma}}$  and  $\Delta_{\text{QED}}$  respectively given by equations (27) and (33). It is useful to rewrite these quantities numerically as

$$\begin{aligned} \frac{\Delta_M}{1 \text{ cm}^{-1}} &= 4 \times 10^{-25} \left( \frac{H_0}{1 \text{ G}} \right) \quad (\text{graviton}), \\ \frac{\Delta_M}{1 \text{ cm}^{-1}} &= 2 \times 10^{-16} \left( \frac{H_0}{1 \text{ G}} \right) \left( \frac{f_{\text{PQ}}}{10^{10} \text{ GeV}} \right)^{-1} \quad (\text{axion}), \\ \frac{\Delta_m}{1 \text{ cm}^{-1}} &= \frac{-2.5 \times 10^{28}}{(2.5 \times 10^{15})^{4/n}} \left( \frac{M_D}{1 \text{ TeV}} \right)^{2+4/n} \left( \frac{\omega}{1 \text{ eV}} \right)^{-1}, \\ \frac{\Delta_{\text{plasma}}}{1 \text{ cm}^{-1}} &= -3.6 \times 10^{-17} \left( \frac{\omega}{1 \text{ eV}} \right)^{-1} \left( \frac{n_e}{1 \text{ cm}^{-3}} \right), \\ \frac{\Delta_{\text{QED}}}{1 \text{ cm}^{-1}} &= 1.33 \times 10^{-27} \left( \frac{\omega}{1 \text{ eV}} \right) \left( \frac{H_0}{1 \text{ G}} \right)^2, \end{aligned} \quad (163)$$

where we have used the facts that  $1 \text{ eV} \simeq 5 \times 10^4 \text{ cm}^{-1}$ ,  $1 \text{ G} \simeq 1.95 \times 10^{-2} \text{ eV}^2$  in the natural Lorentz-Heaviside units where  $\alpha = e^2/4\pi = 1/137$  and the expression of the extra-dimensions radius

$$R = (2.5 \times 10^{15})^{2/n} 10^{-12} \left( \frac{M_D}{1 \text{ TeV}} \right)^{-1-2/n} \text{ eV}^{-1}. \quad (164)$$

We now restrict to the case  $n = 2$ .

### A. Cosmic microwave background

It has been shown that the isotropy of the cosmic microwave background (CMB) puts a limit on the present value of a spatially homogeneous magnetic field to  $(B_0/1\text{G}) \leq 6.8 \times 10^{-9}(\Omega_0 h^2)^{1/2}$  [48,49]. A comparable bound has also been obtained for spatially inhomogeneous magnetic fields [50]. We study the magnitude of the photon-graviton conversion on the two following examples:

- *Large scales:* we assume that we have a homogeneous magnetic field on the scale of the Hubble radius with

$$H_0 \simeq 6 \times 10^{-9} \text{ G}. \quad (165)$$

The CMB photons are observed as a black body with a temperature of 2.7 K so that we approximatively have photons of energy [51,52]

$$\omega \simeq 10^{-5} - 10^{-3} \text{ eV}. \quad (166)$$

The characteristic size of the system is the size of the Hubble radius

$$\Lambda_c = 3000h^{-1} \text{ Mpc} \simeq 10^{28} \text{ cm} \quad (167)$$

where  $h$  is the reduced Hubble parameter. We also estimate the electronic density today to be about (see e.g. [53])

$$n_e \simeq 10^{-7} \text{ cm}^{-3}. \quad (168)$$

- *Degree scales:* we assume a homogeneous magnetic field on the size of the Hubble radius at the last scattering surface. Since the magnetic field scales like (scale factor)<sup>2</sup> and the energy of the photon as (scale factor)<sup>-1</sup>, we assume a magnetic field of

$$H_0 \simeq 6 \times 10^{-3} \text{ G} \quad (169)$$

and consider photons of energy

$$\omega \simeq 10^{-2} - 1 \text{ eV} \quad (170)$$

at a redshift of  $z \simeq 1000$ . The characteristic size of the system is given by the Hubble radius at decoupling, i.e.

$$\Lambda_c = 3 \times 10^{23} h^{-1} \text{ cm}, \quad (171)$$

and the electronic density at the time of decoupling is of order (see e.g. [53])

$$n_e \simeq 10^{-3} \text{ cm}^{-3}. \quad (172)$$

The main idea is that, since photons are converted into either gravitons or axions, some anisotropies must be induced on the scale of homogeneity of the magnetic field, mainly because of the angular dependence of the conversion rate. The effect between a direction parallel and direction perpendicular to the magnetic field must not exceed the observed CMB temperature anisotropy. The anisotropy of the CMB temperature between the directions perpendicular to the magnetic field (where the effect of mixing is maximum) and parallel to it (where there is no mixing effect) is then of order

$$\frac{\Delta T}{T} \simeq \left. \frac{\Delta T}{T} \right|_{\perp} - \left. \frac{\Delta T}{T} \right|_{\parallel} \simeq P(\gamma \rightarrow g). \quad (173)$$

Observationally, we have the constraint [51] that

$$\frac{\Delta T}{T} < 10^{-5}. \quad (174)$$

From figure 3, we deduce that in both cases,  $|\Delta_{\text{QED}}| \ll |\Delta_{\text{plasma}}|$  so that  $\Delta_{\lambda} \simeq \Delta_{\text{plasma}}$  and thus  $\beta > 0$ . In the two considered regimes we have:

1. *Large angular scale:*

$$\alpha \in 6.6 \times [10^{-15}, 10^{-13}]$$

$$\beta \simeq 1.1 \times 10^{21} \left( \frac{M_D}{1 \text{ TeV}} \right)^4. \quad (175)$$

Thus, we are always in a regime where  $\alpha \ll 1$ ,  $\beta > 0$  and  $|\beta| \gg 1$  thus we expect at most effects of order  $\alpha^2$  which are completely unobservable.

2. *Small angular scale:*

$$\alpha \in 6.6 \times [10^{-10}, 10^{-8}]$$

$$\beta \simeq 1.1 \times 10^{17} \left( \frac{M_D}{1 \text{ TeV}} \right)^4. \quad (176)$$

We are always in a regime where  $\alpha^2 \ll 1$ ,  $\beta > 0$  and  $|\beta| \gg 1$  and, as in the previous case, there will be no observable effect.

From this results, with see that  $\beta$  is always too large to have any enhancement of the probability. Moreover in both cases the oscillation length,  $\ell_{\text{osc}}$ , with the lightest KK mode (as well as the oscillation length with any massive KK mode) is much smaller than  $\Lambda_c$ , and the mixing angle with the graviton zero mode is very small. The effects are the same as in a standard four dimensional spacetime and thus negligible [7,8]. Note that, in theory, we should have included the expansion of the universe but this will not change the result drastically. A detailed study of the photon–graviton mixing in an expanding four dimensional spacetime can be found in [6] and a discussion of the effects of the inhomogeneity of the field in [8].

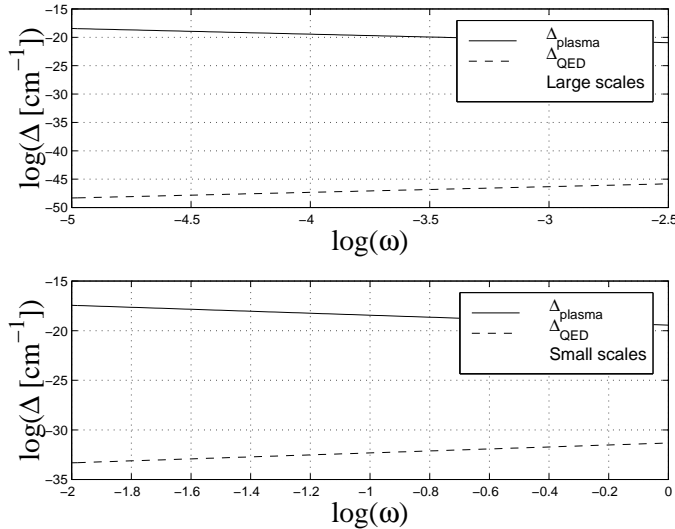


FIG. 3.  $\Delta_{\text{QED}}$  (dash line) and  $|\Delta_{\text{plasma}}|$  (solid line) for the CMB on large (up) and small (down) angular scales. We see that we always have  $\Delta_{\text{QED}} \ll \Delta_{\text{plasma}}$ .

## B. Pulsars

As proposed by many authors (see e.g. [4,14,54]), axions could be produced in the interior of neutron stars in nucleon–nucleon collisions. This would constitute the main cooling mechanism for these stars and thus puts limit on the axion production flux and mass. Such a production of KK gravitons in higher dimensional theories also exist and can be used to put bounds on the mass scale  $M_D$  [30,31].

As originally proposed by Morris [14] (see also [4]), this axion (and now the KK gravitons) flux may be detectable by the secondary photons produced through the mixing with these particles in the neutron star magnetosphere magnetic field. These photons have a typical energy of

$$\omega \simeq 10^4 \text{ eV}, \quad (177)$$

i.e. of order of the average value of the neutron star interior temperature (about 50 keV). The primary photons can be well approximated by a black body spectrum with a temperature of  $T_{NS} \simeq 1 \text{ keV}$  typical for the surface temperature of such stars.

The idea is to detect a distortion of the star spectrum due both to the secondary photons and to the oscillation of primary photons. The typical value of the magnetic field in the neutron star magnetosphere is

$$H_0 \simeq 10^{12} \text{ G} \quad (178)$$

on a characteristic size of the system is of order of the neutron star size

$$\Lambda_c \simeq 10 \text{ km}. \quad (179)$$

Indeed, one cannot neglect the effect of the magnetospheric plasma and we estimate its density [55,56] as

$$n_e \simeq 7 \times 10^{-2} \left( \frac{H_0}{1 \text{ G}} \right) \left( \frac{P}{1 \text{ s}} \right)^{-1} \quad (180)$$

where  $P$  is the period of the pulsar and will be assumed to be about 1 second in the following.

According to figure 4, we deduce that  $\Delta_{\text{QED}} \gg |\Delta_{\text{plasma}}|$  so that  $\beta < 0$  and  $\Delta_\lambda \simeq \Delta_{\text{QED}}$ . Then, it follows

$$\begin{aligned} \alpha &\simeq 3 \times 10^{-14} \\ |\beta| &\simeq 3 \times 10^{-8} \left( \frac{M_D}{1 \text{ TeV}} \right)^4 \end{aligned} \quad (181)$$

and we are always in a regime where  $\alpha \ll 1$ ,  $\beta < 0$  and  $|\beta| \leq 1$  and where the characteristic size of the system is far larger than the oscillation length. We expect  $M_D$  to be of order 1 – 100 TeV, so that we will get an amplification of order 1 –  $10^7$  but still unobservable. From (181) we see that, contrary to the microwave background, the dominant length scale of the system is  $\Delta_{\text{QED}}^{-1}$  so that

$$\frac{\lambda_\gamma}{R} \simeq 2.4 \times 10^{12} \left( \frac{\omega}{1 \text{ eV}} \right)^{-1} \left( \frac{H_0}{1 \text{ G}} \right)^{-1} \left( \frac{M_D}{1 \text{ TeV}} \right)^2 \quad (182)$$

which is smaller than unity for the typical value of magnetic field and wavelength considered here. By going to higher frequencies and higher magnetic fields we may get a larger amplification and thus a larger effect of the extra-dimensions, this is mainly the reason why we will turn to magnetars in the following paragraph.

To finish, let us note however that in very strong magnetic fields one must take into account the photon splitting [3,57] which will compete with the photon-graviton mixing. We do not discuss this effect here.

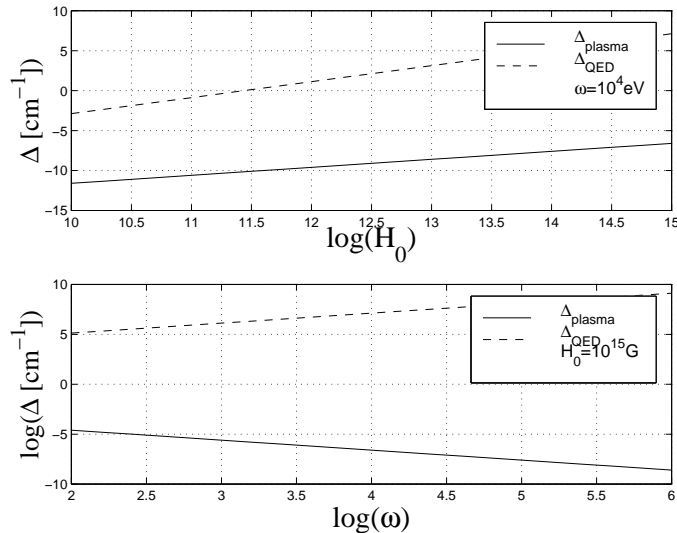


FIG. 4. Variation of  $\Delta_{\text{plasma}}$  (solid line) and  $\Delta_{\text{QED}}$  (dash line) respectively in function of the magnetic field  $H_0$  for a photon of  $\omega = 10^4 \text{ eV}$  (top) and in terms of the frequency for a field of  $H_0 = 10^{15} \text{ G}$  (bottom).

### C. Magnetars and Gamma-Ray Bursts

Magnetars are pulsars with superstrong magnetic field such as SGR 1806-20 [58,59] where  $H_0 \simeq 8 \times 10^{14}$  G, i.e. two orders of magnitude higher than for ordinary radio pulsars. This object is associated with soft gamma ray bursts of energy of order 1 keV – 100 keV. Other examples are GB790305 [60] and IE1841-045 [61] and such observations are supported by models where the gamma ray bursts are triggered by cracking of the neutron star crust due to the magnetic stress [62,63].

So we consider a system such that

$$H_0 \simeq 10^{12} - 10^{15} \text{ G}, \quad \Lambda_c \simeq 10 \text{ km}, \quad (183)$$

and

$$\omega \simeq 10^2 - 10^6 \text{ eV} \quad (184)$$

Assuming that the electronic density is well approximated by (180)<sup>8</sup>, we deduce that we are in the regime  $|\Delta_{\text{plasma}}| \ll \Delta_{\text{QED}}$  as long as

$$\left(\frac{\omega}{1 \text{ eV}}\right)^2 \left(\frac{H_0}{1 \text{ G}}\right) \gg 4 \times 10^9$$

so that we can deduce that the QED contribution always dominates in such object and then that  $\beta < 0$ . On figure 5, we depict the variation of  $\alpha$  to show that we always have  $\alpha \ll 1$ .

Now, effects of the extra-dimensions will appear when  $\lambda_\gamma/R < 1$  and this quantity varies typically from  $10^{-18}$  to  $10^{-25}$  for  $n = 2$  assuming  $M_D \sim 1 \text{ TeV}$  (see equation (182)). Note also that  $\omega^{-1}$  is of the order of  $R$  and that most of the effect comes from the fact that  $\Delta_{\text{QED}}$  becomes large compared with  $R^{-1}$ . Such objects may be interesting to detect the effects of the extra-dimensions but more data and a better understanding of soft gamma-ray bursts are needed before drawing any conclusions.

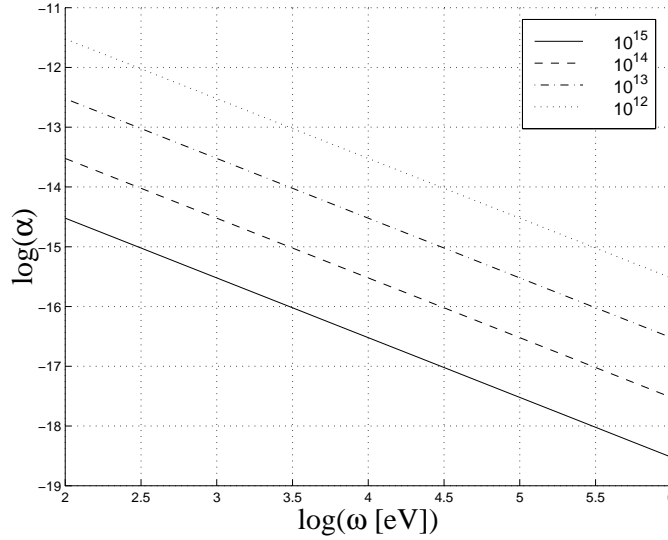


FIG. 5. Variation of  $\alpha$  with respect to the photon frequency for a magnetic field varying between  $H_0 = 10^{12}$  G and  $H_0 = 10^{15}$  G.

### IX. LABORATORY EXPERIMENTS

Many experiments searching for light particles like axions were set up (see e.g. [4,12–15,17,18] and [19] for a recent review). We can classify the methods in the two following categories:

---

<sup>8</sup>The pulsars cited above have a period ranging from 4 to 10 seconds.

- The *direct methods* in which a flux of axions coming from some astrophysical source (Sun, supernovae...) is tried to be converted into photons through an external magnetic field. These transitions have been used to put bounds on astrophysical axion fluxes and coupling constant [12,13]. One could think to use the same kind of experiments to put constraints in the case of a mixing with a large number of KK states. Let us first discuss the case of KK gravitons. The energy flux into KK gravitons from astrophysical object cannot exceed the bound on the energy flux in axions in the usual four dimensional case since otherwise the cooling rates of these objects will be too high [32]. Let us then assume that the efficiency of detection is maximum and that the experiment is designed to collect *all* the emitted particles. Since each graviton is coupled with a much lower coupling constant than the four dimensional axion coupling constant accessible to these kinds of experiments, we do not expect that such *direct* detection methods will be able to see any KK graviton coming from astrophysical sources. In other words, since each KK graviton is coupled at tree level only to the photon (and not to other gravitons), there is no effect of the large number of KK states in these experiments.
- The *indirect methods* where one tries to detect the mixing of the photon through its effect on a photon beam in a magnetic field, both on its amplitude and polarisation. Now, the photon being coupled at tree level to all KK states, one expects a departure from the usual case. The effect on the polarisation of the beam comes from the fact that for axions only the  $\times$  component of electromagnetic wave couples to the axions. For the gravitons, both polarisations evolve according to the same equation but, due to the QED and Cotton–Mouton birefringence,  $\Delta_+ \neq \Delta_\times$  which implies a phase shift between them.

In the next paragraphs, we focus on polarisation experiments to detect the phase shift. We first compute this phase shift in a  $D$  dimensional spacetime and discuss two kinds of experiments respectively in a static and periodic magnetic field. We must emphasize here that the magnitude of the mixing with axions depends on the free parameter  $f_{\text{PQ}}$  (in contrast with the mixing with gravitons which magnitude is fixed by the value of the Planck mass), so that these experiments may be able to put constraints on bulk axion models.

### A. Phase shift in $D$ dimensions

Let us go back to the four dimensional case for axion. Then, from (48), we deduce that, starting from an initial state ( $A(0), G(0) = 0$ ), the two polarisations  $+$  and  $\times$  evolve respectively as, omitting a global phase  $\omega u$ ,

$$A_+(u) = e^{-i\Delta_+ u} A_+(0), \quad A_\times(u) = \left( e^{-i\Delta'_\times u} \cos^2 \vartheta + e^{-i\Delta'_g u} \sin^2 \vartheta \right) A_\times(0). \quad (185)$$

Now, restricting to the weak mixing regime where  $\vartheta \ll 1$ , we can expand the  $\Delta'$  defined in (47) as

$$\Delta'_\times \simeq \Delta_\times + \vartheta^2(\Delta_\times - \Delta_m), \quad \Delta'_g \simeq \Delta_m - \vartheta^2(\Delta_\times - \Delta_m). \quad (186)$$

Expanding (185) to second order and taking into account (186) leads to

$$A_+(u) = e^{-i\Delta_+ u} A_+(0), \quad A_\times(u) = [1 - i\vartheta^2 \zeta + \vartheta^2(e^{i\zeta} - 1)] e^{-i\Delta_\times u} A_\times(0) \quad (187)$$

with  $\zeta \equiv (\Delta_\times - \Delta_m)u$ . We deduce that the two modes evolve relative to each other as

$$\frac{A_\times(u)}{A_+(u)} = [1 - i\vartheta^2 \zeta + \vartheta^2(e^{i\zeta} - 1)] e^{-i(\Delta_\times - \Delta_+)u} \frac{A_\times(0)}{A_+(0)}. \quad (188)$$

The relative phase and amplitude of  $A_\times$  with respect to  $A_+$  then evolve as

$$\left| \frac{A_\times}{A_+} \right| (u) \simeq \left[ 1 - 2\vartheta^2 \sin^2 \left( \frac{\zeta}{2} \right) \right] \left| \frac{A_\times}{A_+} \right| (0), \quad \phi(u) \simeq [(\Delta_+ - \Delta_\times)u - \vartheta^2(\zeta - \sin \zeta)] \phi(0). \quad (189)$$

When we neglect the mixing effect (i.e.  $\vartheta = 0$ ), there is a phase shift due to the QED and Cotton–Mouton birefringence. The extra phase shift due to the fact that only one polarisation of the axion is affected by the mixing has been used to design experiments to put constraints on the axion parameters (see e.g. [4,13,15,64]).

In the case of a graviton, the two polarisations are mixed in the same way, so that the same computation leads to

$$\begin{aligned} \left| \frac{A_\times}{A_+} \right| (u) &\simeq \left( 1 - 2 \left[ \vartheta_\times^2 \sin^2 \left( \frac{\zeta_\times}{2} \right) - \vartheta_+^2 \sin^2 \left( \frac{\zeta_+}{2} \right) \right] \right) \left| \frac{A_\times}{A_+} \right| (0), \\ \phi(u) &\simeq [(\Delta_+ - \Delta_\times)u + \vartheta_+^2(\zeta_+ - \sin \zeta_+) - \vartheta_\times^2(\zeta_\times - \sin \zeta_\times)] \phi(0) \end{aligned} \quad (190)$$

(we now have to keep the index  $\lambda$  on  $\vartheta$  and on  $\zeta$ ). In four dimension,  $\Delta_m = 0$  so that the phase shift depends only on the QED and Cotton–Mouton parameters. Its amplitude is proportional to  $\Delta_M$  so that it is roughly 10 orders of magnitude lower than for axions.

Let us now compute the phase shift in a  $D$  dimensional spacetime. We assume that we are in the weak mixing regime and apply the method of § VII. As seen on (187), we must compute the solution of (149) up to second order. Following the same lines as for the computation leading to (156), we can show that  $\vec{\mathcal{V}}_{\text{int}}^{(2)}$  is explicitly given by

$$\vec{\mathcal{V}}_{\text{int}}^{(2)}(u) = - \sum_q \int_0^u dy \Delta_M(y) \int_0^y dz \Delta_M(z) e^{i \int_y^z [\Delta_m^{(q)} - \Delta_\lambda(x)] dx} A(0) \vec{\mathcal{A}}, \quad (191)$$

From which we deduce that, starting from a pure photon state, the polarisation  $\lambda$  of the photon evolves as

$$A_\lambda(u) = \left[ 1 - \sum_q \int_0^u dy \Delta_M(y) \int_0^y dz \Delta_M(z) e^{i \int_y^z [\Delta_m^{(q)} - \Delta_\lambda(x)] dx} \right] A_\lambda(0). \quad (192)$$

In the case of a homogeneous field, we can extract from (192) the relative phase of the polarisation  $\times$  with respect to the polarisation  $+$  for the case of KK gravitons and bulk axions. In the latter case, only the polarisation  $\times$  evolves according to (192) whereas the polarisation  $+$  evolves according to (187) so that

$$\phi(u) = (\Delta_+ - \Delta_\times)u - \alpha^2 \sum_{i \geq 1} \frac{s_i}{(1 - \beta_i^{(\times)})^2} \left[ (\Delta_\times - \Delta_m^{(r_i)})u - \sin(\Delta_\times - \Delta_m^{(r_i)})u \right] \phi(0) \quad (\text{axion}) \quad (193)$$

$$= (\Delta_+ - \Delta_\times)u - \sum_{i \geq 1} s_i \left\{ \frac{\left[ (\Delta_\times - \Delta_m^{(r_i)})u - \sin(\Delta_\times - \Delta_m^{(r_i)})u \right] \alpha_\times^2}{(1 - \beta_i^{(\times)})^2} - \frac{\left[ (\Delta_+ - \Delta_m^{(r_i)})u - \sin(\Delta_+ - \Delta_m^{(r_i)})u \right] \alpha_+^2}{(1 - \beta_i^{(+)})^2} \right\} \quad (\text{graviton}) \quad (194)$$

with the notations used before. We split this result in three parts as

$$\phi = \phi_{\text{QED}} + \phi_{\text{CM}} + \phi_{KK} \quad (195)$$

where the two first terms are the phase shifts due to vacuum polarisation and the Cotton–Mouton effect and are obtained by setting  $\alpha = 0$  in (193–194). The third term is the specific phase shift associated with the mixing of the photon with either bulk axions or Kaluza–Klein gravitons.

## B. Polarisation experiments

We now discuss two kinds of experiments designed to detect the mixing induced phase shift. As typical parameters we take  $\omega \simeq 2 \text{ eV}$  for the laser beam and a magnetic field which might be as strong as  $H_0 = 10^5 \text{ G}$ . With these values, we have (for  $n = 2$ )

$$\begin{aligned} \frac{\Delta_M}{1 \text{ cm}^{-1}} &\simeq 4 \times 10^{-20}, \quad (\text{graviton}) & \frac{\Delta_M}{1 \text{ cm}^{-1}} &\simeq 2 \times 10^{-11} \left( \frac{f_{\text{PQ}}}{10^{10} \text{ GeV}} \right)^{-1}, \quad (\text{axion}) \\ \frac{\Delta_{\text{QED}}}{1 \text{ cm}^{-1}} &\simeq 3 \times 10^{-17}, \quad \frac{\Delta_{\text{plasma}}}{1 \text{ cm}^{-1}} &\simeq -1.3 \times 10^{-17} \left( \frac{n_e}{1 \text{ cm}^{-3}} \right), & \frac{\Delta_m}{1 \text{ cm}^{-1}} &\simeq -1.5 \times 10^{-4} \left( \frac{M_D}{1 \text{ TeV}} \right)^4. \end{aligned} \quad (196)$$

Then, the QED and plasma effects are of the same order of magnitude but, since  $\phi_{\text{QED}} \gg \phi_{\text{CM}}$ , we neglect the Cotton–Mouton effect and define the phase shift ratio as

$$\mathcal{R}_{KK} \equiv \frac{\phi_{KK}}{\phi_{\text{QED}}}. \quad (197)$$

One hopes to be able to measure a  $\mathcal{R}_{KK}$  of order 0.1 [4].

### 1. Multiple path experiments

The idea is to make a laser beam reflect between two mirrors distant of  $l$ . Since the mirror are transparent to the axions and gravitons, the phase shift after  $N$  paths will be  $N\phi(l)$  so that, in the case of axions,  $\mathcal{R}_{KK}$  is given by

$$\mathcal{R}_{KK} = \left( \frac{\Delta_M}{\Delta_+ - \Delta_-} \right) \sum_{i \geq 1} s_i \frac{\Delta_M}{\Delta_- - \Delta_m^{(r_i)}} \left[ 1 - \frac{\sin(\Delta_- - \Delta_m^{(r_i)})l}{(\Delta_- - \Delta_m^{(r_i)})l} \right]. \quad (198)$$

The decrease of the amplitude of the photon beams due to the creation of axions is

$$\delta I \simeq 4\alpha^2 N \sum_{i \geq 1} \frac{s_i}{(1 - \beta_i)^2} \sin^2 \left[ \frac{\Delta_- - \Delta_m^{(r_i)}}{2} l \right]. \quad (199)$$

Then, an enhancement of both the phase shift and the variation of the beam are expected due to the sums over all states. With the experimental values specified above, we have

$$\frac{\lambda_\gamma}{R} \simeq 10^8 \left( \frac{M_D}{1 \text{ TeV}} \right)^2, \quad (200)$$

which is larger than unity. Then, we think that experiments in homogeneous magnetic fields will not probe the extra-dimensions since it will require to work with very high magnetic fields. Note that when we span the electromagnetic spectrum from the infrared to the X-ray,  $\lambda_\gamma/R$  varies in the range

$$\frac{\lambda_\gamma}{R} \simeq 4 \times (10^8 - 10^2) \left( \frac{M_D}{1 \text{ TeV}} \right)^2. \quad (201)$$

### 2. Effect of a periodic field

As seen in § VII B, one can hope to enhance the mixing effect by using a periodic magnetic field. For that purpose we need the pulsation  $\Delta_0$  to dominate over  $\Delta_\lambda$  and  $\beta$  to be small compared to unity. With the previous numerical values, the first condition rewrites as  $\Delta_0 > 10^{-17} \text{ cm}^{-1}$  and will be satisfied easily. Using (164), the second condition gives for a six dimensional spacetime

$$\Delta_0^{-1} < 2.5 \times 10^2 \left( \frac{M_D}{1 \text{ TeV}} \right)^{-4} \left( \frac{\omega}{1 \text{ eV}} \right) \text{ cm}. \quad (202)$$

As stressed before, we have a departure from the four dimensional behaviour only if  $\lambda_\gamma < R$ . Now, since the magnetic field varies on a scale  $\Delta_0^{-1}$ ,  $\lambda_\gamma$  is governed by  $\Delta_0$  instead of  $\Delta_\lambda$ . Then an effect will appear only if we manage to create a field that can vary on scales of the order of the centimeter.

In § VII B we also quoted the possibility of having a series of strong mixing regimes, which is specific of the existence of extra-dimensions. This requires that  $|\Delta_m^{(r_i)} - (\Delta_\lambda \pm \Delta_0)| \ll \Delta_M \sqrt{s_i}$  and can be performed either by varying  $\Delta_0$  or  $\omega$ . Let us assume that  $\Delta_0$  is fixed, since  $\Delta_0 \gg \Delta_M$  we have a strong mixing regime for the pulsations defined by

$$\omega^{(\vec{p})} = \frac{\vec{p}^2}{2R^2\Delta_0} \quad (203)$$

with a width of

$$\delta\omega^{(\vec{p})} = \frac{\vec{p}^2}{2R^2\Delta_0} \frac{\Delta_M}{\Delta_0}, \quad (204)$$

that is

$$\frac{\omega^{(\vec{p})}}{1 \text{ eV}} = 2 \times 10^{-4} \vec{p}^2 \left( \frac{M_D}{1 \text{ TeV}} \right)^2 \frac{\Delta_0^{-1}}{R}, \quad \frac{\delta\omega^{(\vec{p})}}{1 \text{ eV}} = 10^{-3} \vec{p}^2 \left( \frac{\Delta_0^{-1}}{R} \right)^2 \left( \frac{\Delta_M}{1 \text{ cm}^{-1}} \right). \quad (205)$$

For instance if we assume that  $R$  is of order of the millimeter and that we consider a field varying on the order of the meter, we get for axions that

$$\frac{\omega^{(\vec{p})}}{1 \text{ eV}} = 2 \times 10^{-2} \vec{p}^2 \left( \frac{M_D}{1 \text{ TeV}} \right)^2, \quad \frac{\delta\omega^{(\vec{p})}}{1 \text{ eV}} = 2 \times 10^{-10} \vec{p}^2. \quad (206)$$



## X. CONCLUSION

We have used the property that, in an external magnetic field, the KK gravitons and axions couple at tree level to photons to show that there exists a mixing between the photon and these particles. We have computed this mixing and compared our result to the photon-graviton and photon-axion mixings in four dimensions. The main difference comes from the fact that the mixing matrix is now infinite and that a photon couples to a large number of massive particles. We then have discussed the physical implications of these phenomena in a general  $D$  dimensional universe. This leads us to conclude that for most astrophysical objects the effect of photon-KK gravitons mixing will be unobservable.

The main points of this study are:

- We describe how to deal with the mixing between the photon and a large number of light particles. This extends the former results to the case of  $D$  dimensional universes where a photon can mix with all the Kaluza–Klein gravitons and possibly with bulk axions. In (57–59), we gave the exact expression for the oscillation probability and we then discussed its amplitudes first qualitatively and then in a five and in a six dimensional spacetime.
- When  $\lambda_\gamma < R$ , i.e. if the characteristic length scale associated with the photon wavelength and effective mass is smaller than the radius of the extra–dimensions, there is a departure from the four dimensional effect. Otherwise, the first KK mode is too heavy to be excited and everything, in general, reduces to the four dimensional situation.
- Two limiting regimes have been found:
  - A *large radius regime* where the two following behaviours can appear
    - \* In the *weak mixing* regime, we have shown that the oscillation probability can be obtained by summing over the individual oscillation probabilities and is then enhanced by a factor of order  $\beta^{-n/2}$  in comparison with the standard four dimensional case. In that case the solutions can be found either by solving the eigenvalues equation or by considering the equation of evolution as a Schrödinger equation and solving it iteratively in the interaction picture. The latter method generalises to the case of inhomogeneous magnetic fields but only applies if the oscillation probability is small compared to unity.
    - \* In the *strong mixing* regime the photon mixes preferentially with a given KK modes, which is possible if the plasma effects dominate over the vacuum polarisation. In that case a complete transition is possible. We note that this effect is more likely to happen in a  $D$  dimensional context than in a four dimensional spacetime and point out a specific effect of the extra–dimensions on the oscillation length.
  - A *small radius regime* where  $R \ll \lambda_\gamma$  so that the spacing between to KK modes is very large compared to the photon characteristic length. In that case, the probability is generally dominated by the contribution of the first state corresponding to the lightest particle and we are back to the four dimensional case. A consequence of this is that we can have an observable signature of the extra–dimensions only if  $\lambda_\gamma < R$ . In most of the systems this cannot be achieved, mainly because  $\Delta_\lambda$  is very small; the only favorable situation happens in strong magnetic fields such as in pulsars and magnetars and when we deal with a magnetic field which varies on a small enough typical scale.
- We have shown that, in the case of graviton, the effect of this mixing although enhanced is too small to be observed on the cosmic microwave background and on astrophysical objects such as pulsars. However, we point out that the effects can be larger with bulk axions.
- We discussed laboratory experiments designed for the search of axions in the light of this new framework and we computed the phase shift (193–194) between the two polarisations of a photon entering a magnetic field. As for the probability, we show that the phase shift is enhanced by the existence of the KK modes. In a periodic field, we show that the effect of the extra–dimensions can be important and that there exists a series of strong and weak mixing regimes that may be observed. More work to derive bounds on  $f_{PQ}$  is however needed before drawing any conclusion.
- On a more technical level, we have shown how to diagonalise a general matrix of mixing. This kind of matrices appears in different situations in extra–dimensions physics and these results can be used in many problems and in particular for neutrino oscillations.

- To finish, let us stress some important comments. First, we have assumed the validity of the Euler–Heisenberg Lagrangian to describe the vacuum polarisation. One has also to be aware that in strong magnetic fields such as in pulsar magnetosphere, photon splitting [3,57] will be in competition with the photon graviton oscillation. We have not compared the strength of these two effects but we expect the latter to be dominant in high magnetic fields, such as in the magnetosphere of magnetars. We have also concentrated on gravitational waves even if there is also production of scalar waves. They are thought to be negligible, at least in the four dimensional case [6]. Concerning the axions, the effects may be more important than for gravitons, depending on the value of the coupling  $f_{PQ}$ , but we did not reconsider the bounds on the axion parameters. Further work is needed in that direction. We also stress that in general one needs a precise determination of their mass spectrum to compute the coupling of each mode to the photon. Most of these results were obtained for  $n = 1$  or  $n = 2$  extra-dimensions for which the results respectively do not, or only weakly, depend on the cut-off of the theory. We also stress that depending on the exact physical situation the number of KK modes with which the photon can oscillate coherently can be drastically limited in comparison with the number of accessible KK modes. These decoherence effect implies that the UV cut-off of the theory  $M_{\text{max}}$  is expected to be much higher than the physical cut-off. A consequence of this is that the results obtained in the cases  $n = 1$  and  $n = 2$  can be extended to  $n > 2$  without depending on  $M_{\text{max}}$ .

## ACKNOWLEDGMENTS

We wish to thank l'École de Physique Théorique des Houches where this work was initiated, P. Binétruy, E. Dudas, J.F. Glicenstein, R. Lehoucq, M. Lemoine, J. Mourad, O. Pene, P. Peter and J. Rich for discussions.

- 
- [1] M.E. Gertsenshtein, Sov. Phys. JETP **14** (1962) 84.
  - [2] Ya. B. Zel'dovich and I.D. Novikov, *The structure and evolution of the universe*, Relativistic Astrophysic Vol. 2 (Chicago University Press) (1983).
  - [3] S.L. Adler, Ann. Phys. (N.Y.) **67** (1971) 599.
  - [4] G.G. Raffelt and L. Stodolsky, Phys. Rev. **D37** (1988) 1237.
  - [5] L. Landau and E. Lifschitz, *Théorie des champs* (Eds MIR, Moscou), Quatrième édition (1989).
  - [6] J.C.R. Magueijo, Phys. Rev. **D49** (1994) 671.
  - [7] P. Chen, Phys. Rev. Lett. **74** (1995) 634.
  - [8] A.N. Cillis and D. Harari, Phys. Rev. **D54**(1996) 4757.
  - [9] J.E. Kim, Phys. Rep. **150** (1987) 1.
  - [10] G.G. Raffelt, Phys. Rep. **198** (1990) 1.
  - [11] M. Turner, Phys. Rep. **197** (1990) 67.
  - [12] P. Sikivie, Phys. Rev. Lett. **51** (1983) 1415.
  - [13] P. Sikivie, Phys. Rev. **E52** (1984) 695.
  - [14] D.E. Morris, Phys. Rev. **D34** (1986) 843.
  - [15] K. van Bibber *et al.*, Phys. Rev. **D39** (1989) 2089.
  - [16] C. Hagmann *et al.* Phys. Rev. Lett. **80** (1998) 2043.
  - [17] J.E. Kim, Phys. Rev. **D58** (1998) 055006.
  - [18] P. Sikivie, Phys. Lett. **B432** (1998) 139.
  - [19] G.G. Raffelt, Proc. of Neutrino '98, Japan, Y. Suzuki and Y. Totsuka eds. (1998).
  - [20] N. Arkani-Hamed, S. Dimopoulos and G.Dvali, Phys. Lett. **B429** (1998) 263.
  - [21] N. Arkani-Hamed, S. Dimopoulos and G.Dvali, Phys. Rev. **D59** (1999) 086004.
  - [22] E. Witten, Nucl. Phys. **B471** (1996) 135.
  - [23] J.D. Lykken, Phys. Rev. **D54** (1996) 3693.
  - [24] K.R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. **B436** (1998) 55; *ibid* Nucl. Phys. **B537** (1999) 47.
  - [25] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B436** (1998) 257.
  - [26] I. Antoniadis, Phys. Lett. **B246** (1990) 377; I. Antoniadis, C. Muñoz and M. Quirós, Nucl. Phys. **B397** (1993) 515; I. Antoniadis, K. Benakli, Phys. Lett. **B326** (1994) 69; I. Antoniadis, K. Benakli and M. Quirós, Phys. Lett **B331** (1994) 313; A. Pomarol and M. Quirós, [[hep-ph/9806263](#)].
  - [27] G.F. Giudice, R. Rattazzi and J.D. Wells, Nucl. Phys. **B544** (1999) 3.
  - [28] T. Han, J.D. Lykken and R.-J. Zhang, Phys. Rev. **D59** (1999) 105006.

- [29] P. Binétruy, C. Deffayet and D. Langlois, [[hep-th/9905012](#)]; P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, [[hep-th/9910219](#)] and references therein.
- [30] V. Barger, T. Han, C. Kao and R.-J. Zhang, Phys.Lett. **B461** (1999) 34.
- [31] S. Cullen and M. Perelstein, Phys. Rev. Lett. **83** (1999) 268.
- [32] S. Chang, S. Tazawa and M. Yamaguchi, [[hep-ph/9908515](#)]; *ibid.*, [[hep-ph/9909240](#)].
- [33] K.R. Dienes, E. Dudas and T. Gherghetta, [[hep-ph/9912455](#)].
- [34] E. Dudas and J. Mourad, [[hep-th/9911019](#)].
- [35] J. Rich, Phys. Rev **D48** (1993) 4318; C.Y. Cardall and D.J.H. Chung [[hep-ph/9904291](#)]; B. Kayser, Phys. Rev. **D24** (1981) 110; W. Grimus and D. Stockinger, Phys. Rev. **D54** (1996) 3414.
- [36] W. Heisenberg and H. Euler, Z. Phys **88** (1936) 714.
- [37] C. Itzykson and J-B. Zuber, *Quantum field theory*, MacGraw-Hill, New York (1980).
- [38] L.D. Landau and E.M. Lifschitz, *Electrodynamics of continuous media*, MIR (1960).
- [39] *American Institute of Physics Handbook*, 3rd. ed., D.E. Gray Eds, McGraw-Hill, New York (1972).
- [40] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and J-M. Russel, [[hep-ph/9811448](#)].
- [41] K.R. Dienes, E. Dudas and T. Gherghetta, [[hep-ph/9811428](#)].
- [42] L. Randall and R. Sundrum, [[hep-ph/9905221](#)]; *ibid*, [[hep-th/9906064](#)].
- [43] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals series and product* Ed. Academic, N.Y. (1990).
- [44] P.P. Kronberg, Rep. Prog. Phys. **57** (1994) 325.
- [45] K. Enqvist, Int. J. Mod. Phys. **D7** (1998) 331.
- [46] T.W.B. Kibble and A. Vilenkin, Phys. Rev. **D56** (1995) 679.
- [47] M.S. Turner and L.M. Widrow, Phys. Rev. **D37** (1988) 2743.
- [48] J.D. Barrow, P.G. Ferreira and J. Silk, Phys. Rev. Lett. **78** (1997) 3610.
- [49] D.Puy and P. Peter, Int. J. Mod. Phys. **D7** (1998) 489.
- [50] K. Subramanian and J.D. Barrow, Phys. Rev. Lett. **81** (1998) 3575.
- [51] G.F. Smoot *et al.*, Astrophys. J. **396** (1992) L1.
- [52] J.C. Mather *et al.*, Astrophys. J. **354** (1990) L37.
- [53] P.J.E. Peebles, *Principles of Physical Cosmology* Princeton University Press (1993).
- [54] N. Iwamoto, Phys. Rev. Lett. **53** (1984) 1198.
- [55] L. Shapiro and S.A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars*, Wiley, New-York (1983).
- [56] P. Goldreich and W.H. Julian, Astrophys. J. **157** (1969) 869.
- [57] S.L. Adler, J.N. Bahcall, C.G. Callan and M.N. Rosenbluth, Phys. Rev. Lett. **25** (1970) 1061.
- [58] C. Kouveliotou *et al.*, Nature **393** (1998) 235.
- [59] C. Kouveliotou *et al.*, to appear in Astrophys. J. Lett., [[astro-ph/9809140](#)]
- [60] B. Paczyński, Acta Astron. **42(3)** (1992) 145.
- [61] G. Vasisht and E. Gotthelf, Astrophys. J. **486** (1997) L129.
- [62] C. Thompson and R. Duncan, Month. Not. R. Astron. Soc. **275** (1995) 255.
- [63] C. Thompson and R. Duncan, Astrophys. J. **473** (1996) 322.
- [64] L. Maiani, R. Petronzio and E. Zavattini, Phys. Lett. **B175** (1986) 359.

## APPENDIX A: DIAGONALISATION OF $\mathcal{M}$ IN THE GENERAL CASE

The goal of this appendix is to compute the eigenvalues and eigenvector of the matrix  $\mathcal{M}$ , to diagonalise it and to explain how to compute the probability of conversion of a photon into a graviton.

We consider the  $(N+2) \times (N+2)$  matrix defined by

$$\mathcal{M} = \begin{pmatrix} \Delta_\lambda & \Delta_M & \cdots & \cdots & \cdots & \Delta_M \\ \Delta_M & \Delta_m^{(0)} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \Delta_m^{(q)} & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \Delta_M & 0 & \cdots & \cdots & 0 & \Delta_m^{(N)} \end{pmatrix}. \quad (\text{A1})$$

We restrict to a finite matrix since there is a cut-off in the theory as discused in § IV B and all the notations are defined in § II.

Let us stress that the diagonalisation of this matrice is a purely technical point that appears often in extra-dimension physics (see e.g. [41]).

### 1. Characteristic polynomial

These notations being fixed, we compute the characteristic polynomial  $\mathcal{P}(x)$  of the matrix  $\mathcal{M}$  defined by

$$\mathcal{P}(x) \equiv \det(\mathcal{M} - xI_{N+2}), \quad (\text{A2})$$

where  $I_{N+2}$  is the  $(N+2) \times (N+2)$  identity matrix. Developping (A2) with respect to its first column leads to

$$\mathcal{P}(x) = (\Delta_\lambda - x) \prod_{q=0}^N (\Delta_m^{(q)} - x) + \sum_{q=0}^N (-1)^{q+1} \Delta_M D_q(x), \quad (\text{A3})$$

where  $D_q$  is the determinant of the comatrix of the element  $(q+2, 1)$  given by

$$D_q(x) = (-1)^q \Delta_M \prod_{\ell=0, \ell \neq q}^N (\Delta_m^{(\ell)} - x). \quad (\text{A4})$$

From (A3) and (A4), we deduce that the characteristic polynomial is given by

$$\mathcal{P}(x) = \prod_{q=0}^N (\Delta_m^{(q)} - x) \left[ \Delta_\lambda - x - \Delta_M^2 \sum_{q=0}^N \frac{1}{(\Delta_m^{(q)} - x)} \right]. \quad (\text{A5})$$

### 2. Eigenvalues

The characteristic eigenvalue equation  $\mathcal{P}(x) = 0$  has  $N+2$  real solutions since  $\mathcal{M}$ , being a symetric matrix, is diagonalisable. To find all his solutions, we rewrite (A5) as

$$\mathcal{P}(x) = A(x)B(x), \quad (\text{A6})$$

with

$$A(x) = \prod_{i=1}^{N_D} (\Delta_m^{(r_i)} - x)^{s_i-1}$$

$$B(x) = (\Delta_\lambda - x) \prod_{i=1}^{N_D} (\Delta_m^{(r_i)} - x) - \Delta_M^2 \sum_{i=1}^{N_D} \prod_{\ell=1, \ell \neq i}^{N_D} s_\ell (\Delta_m^{(r_\ell)} - x). \quad (\text{A7})$$

We have two kind of eigenvalues:

- $\Delta_m^{(r_i)}$ : they are solutions of  $A(x) = 0$  and are of order  $s_i - 1$  but are not solutions of  $B(x) = 0$  so that they are solutions of  $\mathcal{P}(x) = 0$  with order  $s_i - 1$  and thus are eigenvalues of  $\mathcal{M}$  of the same order.

This gives us  $\sum_{i=1}^{N_D} (s_i - 1) = N + 1 - N_D$  eigenvalues of  $\mathcal{M}$ .

- $x_i$ : they are solutions of  $B(x) = 0$  and since  $B(x)$  is a polynomial of order  $N_D + 1$  and since  $\mathcal{M}$  is diagonalisable, we must have  $N_D + 1$  such eigenvalues. To find them, we rewrite  $B(x)$  as

$$B(x) = \left[ \prod_{i=1}^{N_D} (\Delta_m^{(r_i)} - x) \right] \left[ \Delta_\lambda - x - \Delta_M^2 \sum_{j=1}^{N_D} \frac{s_j}{(\Delta_m^{(r_j)} - x)} \right]. \quad (\text{A8})$$

Let us stress that  $x_i \neq \Delta_m^{(r_i)}$  since otherwise the cancellation occuring in the first factor is offset by a divergence in the second factor. It follows that the  $x_i$  are solutions of

$$\Delta_\lambda - x = \Delta_M^2 \sum_{i=1}^{N_D} \frac{s_i}{(\Delta_m^{(r_i)} - x)}. \quad (\text{A9})$$

This solution can be found numerically but we can find the main properties of these eigenvalues graphically from which we deduce that (A9) has  $N_D + 1$  *distinct* solutions that we order as

$$(x_i)_{1 \leq i \leq N_D+1} \quad x_1 < \dots < x_{N_D+1}. \quad (\text{A10})$$

In conclusion, we have found the  $N + 2$  eigenvalues of  $\mathcal{M}$  which split in  $n$  eigenvalues  $\Delta_m^{(r_i)}$  each with multiplicity  $s_i - 1$  and in  $N_D + 1$  distinct eigenvalues  $x_i$ .

### 3. Eigenvectors

To determine the eigenvectors  $V$  solution of

$$\mathcal{M}V = xV, \quad (\text{A11})$$

we set

$$V \equiv (v, u_0, \dots, u_N) \quad (\text{A12})$$

so that (A11) reduces to the system

$$\Delta_\lambda v + \Delta_M \left( \sum_{q=0}^N u_q \right) = xv \quad (\text{A13})$$

$$\Delta_M v + \Delta_m^{(q)} u_q = xu_q. \quad (\text{A14})$$

- If  $x = \Delta_m^{(r_i)}$ , the eigenvectors generate a subspace of dimension  $s_i - 1$  a basis of which is given explicitly by

$$V_{r_i+p} = \frac{1}{\sqrt{(p+1)(p+2)}} \left[ - \sum_{\ell=0}^p G_{r_i+\ell} + (p+1)G_{r_i+p+1} \right], \quad 0 \leq p \leq s_i - 2, \quad (\text{A15})$$

where  $\{A, (G_q)_{0 \leq q \leq N}\}$  is the initial orthonormal basis where we have written  $\mathcal{M}$  in (A1). One can check that this is an orthonormal family, i.e. that

$$\langle V_{r_i+p} | V_{r_j+q} \rangle = \delta_{pq} \delta_{ij}.$$

- If  $x = x_i$ , for each eigenvalue we have a subspace of dimension 1 generated by the unit vector

$$V_{x_i} = \frac{1}{\sqrt{\Delta_M^{-2} + \sum_{q=0}^N \left(x_i - \Delta_m^{(q)}\right)^{-2}}} \left( \frac{1}{\Delta_M}, \frac{1}{x_i - \Delta_m^{(0)}}, \dots, \frac{1}{x_i - \Delta_m^{(N)}} \right) \quad (\text{A16})$$

in the basis  $\{A, (G_q)_{0 \leq q \leq N}\}$ . It is easy to show that they satisfy

$$\langle V_{x_i} | V_{x_j} \rangle = \delta_{ij}, \quad \langle V_{x_i} | V_{r_j+p} \rangle = 0.$$

We have given the explicit form of the  $N + 2$  eigenvectors of  $\mathcal{M}$ . It is worthwhile noting that the eigenstates  $V_{r_i+p}$  mix the different KK modes together while letting the photon unaffected whereas the eigenstates  $V_{x_i}$  mix the photon with the  $N + 1$  KK gravitons.

## APPENDIX B: PROBABILITY OF OSCILLATION IN THE GENERAL CASE

To compute the oscillation probability between a photon and gravitons in a constant magnetic field, we follow the method by Raffelt and Stodolsky [4] and solve the equation of evolution (25) as

$$\vec{\mathcal{V}}(u) = e^{-i\mathcal{M}u} e^{-i\omega u} \vec{\mathcal{V}}(0), \quad (\text{B1})$$

where  $\vec{\mathcal{V}} \equiv \{A, G^{(0)}, \dots, G^{(N)}\}$ . We decompose this vector on the eigenvectors basis as

$$\vec{\mathcal{V}}(0) = \sum_{i=1}^{N_D} \sum_{p=0}^{s_i-2} h_{i,p}(0) V_{r_i+p} + \sum_{i=1}^{n+1} f_i(0) V_{x_i}, \quad (\text{B2})$$

where  $h_{i,p}(0)$  and  $f_i(0)$  are  $N + 2$  coefficients. Injecting this decomposition in (B1), we obtain

$$\vec{\mathcal{V}}(z) = \left[ \sum_{i=1}^{N_D} \sum_{p=0}^{s_i-2} h_{i,p}(0) e^{-i\Delta_m^{(r_i)} u} V_{r_i+p} + \sum_{i=1}^{N_D+1} f_i(0) e^{-ix_i u} V_{x_i} \right] e^{-i\omega u}. \quad (\text{B3})$$

The probability of a photon to be converted in KK gravitons is obtained by considering the initial state  $\vec{\mathcal{V}}(0) = \{1, 0, \dots, 0\}$  and by computing

$$P(\gamma \rightarrow \gamma) = \left| \sum_{q=0}^N \langle G^{(q)}(z) | \vec{\mathcal{V}}(0) \rangle \right|^2. \quad (\text{B4})$$

Since only the modes associated with the eigenvalues  $x_i$  mix with the photon, we deduce that

$$P(\gamma \rightarrow g) = 1 - P(\gamma \rightarrow \gamma) = 1 - \left| \sum_{i=1}^{N_D+1} f_{x_i}^2 e^{ix_i u} \right|^2, \quad (\text{B5})$$

where the coefficients  $f_{x_i}$  are

$$f_{x_i} = \left[ 1 + \sum_{q=0}^N \frac{\Delta_M^2}{\left(x_i - \Delta_m^{(q)}\right)^2} \right]^{-1/2}. \quad (\text{B6})$$

## APPENDIX C: UPPER BOUND ON $|\mathcal{F}_J|$ AND $|\mathcal{G}_J|$

The goal of this appendix is to give an upper bound on the absolute value of the two functions  $\mathcal{F}_J(y)$  and  $\mathcal{G}_J(y)$  (defined in (109) and (C16), (C17)) when  $n = 2$ . These majorations are used in section VI to determine the solution of the eigenvalues equation (108) as well as the oscillation probability (57) in a small coupling limit. We also give an upper bound on  $s_i$ . We first give a bound on

$$\mathcal{F}_{i_1, i_2}(y) \equiv \alpha^2 \sum_{i=i_1}^{i_2} \frac{s_i}{y - \beta_i}, \text{ and on} \quad (\text{C1})$$

$$\mathcal{G}_{i_1, i_2}(y) \equiv \alpha^2 \sum_{i=i_1}^{i_2} \frac{s_i}{(y - \beta_i)^2}. \quad (\text{C2})$$

We recall that the  $\beta_i$  are defined by  $\beta_i \equiv \vec{p}_i^2 \beta \equiv p_i^2 \beta$ , where  $\vec{p}_i$  is a pair  $(n_i, m_i)$  of integers, and  $p_i$  is defined by the second equality. We assume all along this discussion that  $\beta$  is positive and we order the  $\beta_i$  as  $\beta_i < \beta_{i+1}$ . We stress that  $\beta_1 = 0$ .

$$\mathbf{1.} \quad 0 \leq y < \beta_{i_1} < \beta_{i_2}$$

For  $i \geq i_1$ , each  $\vec{p}_i^2$  belongs to a unique interval

$$(p_{i_1} + k)^2 \leq p_i < (p_{i_1} + k + 1)^2, \quad (\text{C3})$$

where  $k$  is an integer.  $\mathcal{M}_k$ , the number of such  $\vec{p}_i$ , is bounded by four times the surface defined by

$$p_{i_1} + k \leq \sqrt{x_1^2 + x_2^2} \leq p_{i_1} + k + 1 \quad (\text{C4})$$

in the real plan  $(x_1, x_2)$ , since we have at most four pairs for each unit square cell. Thus, one has

$$\mathcal{M}_k \leq 8\pi(p_{i_1} + k + 1). \quad (\text{C5})$$

One can then easily obtain a majoration in term of  $(p_{i_1} + k)$ :

$$\mathcal{M}_k \leq q(p_{i_1} + k), \quad (\text{C6})$$

with  $q = 16\pi$ . On the other hand, for each  $p_i$  satisfying (C3),  $|1/(y - \beta_i)|$  is lower than  $|1/(y - (p_{i_1} + k)^2 \beta)|$ , from which we get the upper bound on  $\mathcal{F}_{i_1, i_2}(y)$

$$|\mathcal{F}_{i_1, i_2}(y)| \leq \frac{\alpha^2 q}{\beta} \sum_{k=0}^{k_{\max}} \frac{\sqrt{\beta}(p_{i_1} + k)}{(\sqrt{\beta}(p_{i_1} + k))^2 - y} \sqrt{\beta}. \quad (\text{C7})$$

$k_{\max}$  is defined such that  $p_{i_1} + k_{\max} < p_{i_2} < p_{i_1} + k_{\max} + 1$ . The r.h.s. of (C7) is nothing else but a Riemann sum associated with the function  $f(x) = x/(x^2 - y)$ . Since  $f(x)$  is decreasing for all  $x^2 > y$ , we have

$$|\mathcal{F}_{i_1, i_2}(y)| \leq \frac{\alpha^2 q}{\beta} \int_{\sqrt{\beta}(p_{i_1}-1)}^{\sqrt{\beta}(p_{i_1}+k_{\max})} \frac{x dx}{x^2 - y} \leq \frac{\alpha^2 q}{2\beta} \ln \left( \frac{\beta(p_{i_1} + k_{\max})^2 - y}{\beta(p_{i_1} - 1)^2 - y} \right). \quad (\text{C8})$$

Let us emphasize that (C8) assumes implicitly that  $y < \beta(p_{i_1} - 1)^2$ . Otherwise, the contribution of the  $\beta_i$  such that  $\beta p_{i_1}^2 \leq \beta_i < \beta(p_{i_1} + 1)^2$  in the sum (C1) can be bounded by

$$\frac{\alpha^2 q}{\beta p_{i_1}^2 - y} p_{i_1}. \quad (\text{C9})$$

Since  $\beta(p_{i_1} - 1)^2 \leq y < \beta p_{i_1}^2$ , we deduce that

$$p_{i_1} \leq \left( \frac{\sqrt{y}}{\sqrt{\beta}} + 1 \right), \quad (\text{C10})$$

so that we get the majoration

$$|\mathcal{F}_{i_1, i_2}(y)| \leq \frac{\alpha^2 q}{(\beta p_{i_1}^2 - y)} \left( \frac{\sqrt{y}}{\sqrt{\beta}} + 1 \right) + \frac{\alpha^2 q}{\beta} \ln \left( \frac{\beta(p_{i_1} + k_{\max})^2 - y}{\beta p_{i_1}^2 - y} \right). \quad (\text{C11})$$

With similar arguments, one can show that, when  $y < \beta(p_{i_1} - 1)^2$ ,  $|\mathcal{G}_{i_1, i_2}(y)|$  is bounded by

$$|\mathcal{G}_{i_1, i_2}(y)| \leq \frac{\alpha^2 q}{\beta} \int_{\sqrt{\beta}(p_{i_1} - 1)}^{\sqrt{\beta}(p_{i_1} + k_{\max})} \frac{xdx}{(x^2 - y)^2} \leq \frac{\alpha^2 q}{2\beta} \left( \frac{1}{\beta(p_{i_1} - 1)^2 - y} - \frac{1}{\beta(p_{i_1} + k_{\max})^2 - y} \right), \quad (\text{C12})$$

and that otherwise

$$|\mathcal{G}_{i_1, i_2}(y)| \leq \frac{\alpha^2 q}{(\beta p_{i_1}^2 - y)^2} \left( \frac{\sqrt{y}}{\sqrt{\beta}} + 1 \right) + \frac{\alpha^2 q}{2\beta} \left( \frac{1}{\beta p_{i_1}^2 - y} - \frac{1}{\beta(p_{i_1} + k_{\max})^2 - y} \right). \quad (\text{C13})$$

The bounds (C11) and (C13) are valid also in the case where  $y < \beta(p_{i_1} - 1)^2$ .

## 2. $0 \leq \beta_{i_1} < \beta_{i_2} < y$

In that case, following the same line of reasoning, we obtain respectively for  $\mathcal{F}_{i_1, i_2}$  and  $\mathcal{G}_{i_1, i_2}$  and any  $y$  satisfying the above condition

$$|\mathcal{F}_{i_1, i_2}(y)| \leq \frac{\sqrt{y}\alpha^2 q}{\sqrt{\beta}(y - \beta p_{i_2}^2)} + \frac{\alpha^2 q}{2\beta} \ln \left( \frac{y - \beta(p_{i_2} - k_{\max})^2}{y - \beta p_{i_2}^2} \right), \quad (\text{C14})$$

$$|\mathcal{G}_{i_1, i_2}(y)| \leq \frac{\sqrt{y}\alpha^2 q}{\sqrt{\beta}(y - \beta p_{i_2}^2)^2} + \frac{\alpha^2 q}{\beta} \left( \frac{1}{y - \beta p_{i_2}^2} - \frac{1}{y - \beta(p_{i_2} - k_{\max})^2} \right), \quad (\text{C15})$$

## 3. bound on $\mathcal{F}_J$ and $\mathcal{G}_J$

We now give a bound on the expression defined by

$$\mathcal{F}_J(y) \equiv \alpha^2 \sum_{i=1, i \neq J}^{N_D} \frac{s_i}{y - \beta_i}, \quad (\text{C16})$$

$$\mathcal{G}_J(y) \equiv \alpha^2 \sum_{i=1, i \neq J}^{N_D} \frac{s_i}{(y - \beta_i)^2}, \quad (\text{C17})$$

For  $J$  defined by

$$\forall i \neq J, \quad |y - \beta_i| \geq |y - \beta_J|. \quad (\text{C18})$$

One has thus  $\forall i \neq J$ ,  $|y - \beta_i| \geq \beta/2$ , and in particular  $\beta p_{J+1}^2 - y > \beta/2$  and  $y - \beta p_{J-1}^2 > \beta/2$ . Using equations (C11–C15) as well as the value of  $p_{\max}$  given<sup>9</sup> in (31), one obtains easily the following bound on  $|\mathcal{F}_J|$  and  $|\mathcal{G}_J|$ <sup>10</sup>.

---

<sup>9</sup>Note that the energy cut-off  $p_{\max}$  has to be regarded as a maximum one fixed by the underlying quantum regularisation of the theory. However, decoherence effects can reduce drastically the number of KK modes that have to be taken into account. Thus, the bounds on  $\mathcal{Q}$  and  $\mathcal{Q}'$  in a more realistic case are likely to be lower than the one given here.

<sup>10</sup>Similar bounds would also apply to  $\mathcal{F}(y)$  and  $\mathcal{G}(y)$  when  $y$  is so that  $\forall i \in [1, N_D]$ ,  $|y - \beta_i| \geq \beta/2$ .



$$|\mathcal{F}_J| \leq \mathcal{Q} \frac{\alpha^2}{\beta^2} \sup \left( \sqrt{\beta y}, \mathcal{Q}' \right) \quad (\text{C19})$$

$$|\mathcal{G}_J| \leq \mathcal{Q} \frac{\alpha^2}{\beta^2} \sup \left( \frac{\sqrt{y}}{\sqrt{\beta}}, 2 \right) \quad (\text{C20})$$

where  $\mathcal{Q}$  is a constant of order  $10^3$  and  $\mathcal{Q}'$  a constant of order 10.

#### 4. bound on $s_i$

In the real plane the euclidian distance between two different pairs of integers is bounded by 1, so that the number of pairs of integers on a given closed curve is always lower than the length of this curve. It is then easy to obtain the following bound on  $s_i$  which represents then number of different pairs of integers on a circle of radius  $p_i$ .

$$s_i \leq 2\pi p_i = 2\pi \sqrt{\frac{\beta_i}{\beta}}. \quad (\text{C21})$$

To finish, let us note that the properties of the series  $s_i$  have been studied by Gauss around 1800, see e.g. [<http://mathworld.wolfram.com/rn.html>] for details and references.